A Cell-Centered Scheme For Heterogeneous Anisotropic Diffusion Problems On General Meshes

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Abstract

We present a new scheme for the discretization of heterogeneous anisotropic diffusion problems on general meshes. With light assumptions, we show that the algorithm can be written as a cell-centered scheme with a small stencil and that it is convergent for discontinuous tensors. The key point of the proof consists in showing both the strong and the weak consistency of the method.
The efficiency of the scheme is demonstrated through numerical tests of the 5th International Symposium on Finite Volumes for Complex Applications - FVCA 5. Moreover, the comparison with classical finite volume schemes emphasizes the precision of the method. We also show the good behaviour of the algorithm for nonconforming meshes.

Key words : Heterogeneous anisotropic diffusion, general grids, finite volumes, finite elements, cell-centered scheme.

1 Introduction

The approximation of the solutions of the anisotropic heterogeneous diffusion problems is an important issue in several engineering fields. We mention a kind of these problems, as follows:

$$\begin{cases} -\mathrm{div}\,(\Lambda(x)\nabla u) \;\; = f \qquad \text{in } \Omega, \\ u = 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the following assumptions hold:

(a) Ω is an open bounded connected polygonal subset of $\mathbb{R}^2$.

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(b) The diffusion (or permeability) tensor $\Lambda : \Omega \to \mathbb{R}^{2\times 2}$ is symmetric, piecewise Lipschitz-continuous and such that the set of its eigenvalues is included in $[\underline{\lambda}, \overline{\lambda}]$, with $\underline{\lambda}$ and $\overline{\lambda} \in \mathbb{R}$ satisfying $0 < \underline{\lambda} \leq \overline{\lambda}$.

(c) The function f is the source term and belongs to $L^2(\Omega)$.

With assumptions (a)-(c), u is called the w solution of (1) if u satisfies

$$u \in H_0^1(\Omega) \text{ and } \forall v \in H_0^1(\Omega), \int_\Omega (\Lambda(x)\nabla u(x)) .\nabla v(x)dx = \int_\Omega f(x)v(x)dx. \quad (2)$$

The list of well-known discretization methods are finite difference methods, finite element methods and finite volume methods. Each of these methods has its own advantages and disadvantages.
The finite difference method is simple. Nevertheless, it is not widely used in practical applications, since we need a smoothness assumption of the solution and it is not applicable for the domains with a complex geometry.
The standard finite element method has the following advantages:

- It can be applied in domains with complex shapes. These domains can be discretized by using triangular meshes.
- It uses the spaces of piecewise polynomials of degree 1 to approximate the solution function. The basic functions of these spaces have small supports, so the computation of this method is simple.

Unfortunately, in discontinuous diffusion problems coupled with convective transport models, the approximation solutions computed by the standard finite element method can be inaccurate [32].
The finite volume method which is a popular discretization method, is used to approximate the solutions of anisotropic heterogeneous diffusion problems. It allows us to obtain the local conservativity of the fluxes which is significant in physics. This method is classified into two main categories:

- "Cell-centered schemes" compute approximation values of the solution function at the centers of the cells of the primary mesh.
- Other schemes use not only usual cell unknowns but also interface unknowns to compute approximation values of the solution function. In [14], HFV, MFD and MFV which involve the cell and edge unknowns are schemes of the same family. Besides, the DDFV schemes in [11], [21] which give precise solutions, use techniques of dual mesh and involve the cell and vertex unknowns. However, all schemes in this category are computationally more expensive than cell-centered schemes, because they use more unknowns.

Therefore, we pay attention to "Cell-centered schemes" which have small stencils and only use cell unknowns. The so-called Multi Point Flux Approximation (MPFA) [1], [2] involves the reconstruction of the gradient in order to evaluate the fluxes. Nevertheless, these methods only satisfy coercivity under suitable conditions on

both the mesh and the permeability tensor Λ. In [5], the authors need a coercivity assumption which links the mesh and the tensor. There are also some schemes which need either conditions on meshes or conditions on the permeability tensor. For example, in [3], the condition is that the meshes are not too distorted. In [10], the authors need a sufficient coercivity condition. In this kind of schemes, let us also cite methods [12, 26, 27, 28, 29, 30, 31] which allow to obtain maximum and minimum principles for diffusion problems on distorted meshes.
In this paper, the proposed scheme is designed on general meshes for heterogeneous and anisotropic permeability tensors with the following advantages:

- The main idea of the scheme is based on that of the standard finite element method and uses a technique of dual mesh. The dual mesh is chosen to be easily recovered a cell centered scheme, i.e, the dual mesh unknowns are computed by linear combinations of cell unknowns. This is different from other schemes which use techniques of dual mesh such as DDFV schemes, because we can not recover a cell centered scheme when using the dual mesh of the DDFV scheme.

- It is a cell-centered scheme and its stencil is equal or less than a nine point stencil on two dimensional quadrangular meshes.

- In heterogeneous and homogeneous anisotropic cases, it is locally conservative.

- In general cases, using a light assumption (hypothesis 3.1), the matrix which is associated to our scheme, is symmetric and positive definite on general meshes.

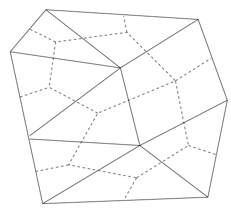
The work is organized as follows: in §2, we introduce the methods to construct a dual grid and a third grid. In §3, we present the scheme in isotropic homogeneous case and in anisotropic heterogeneous case. Besides, we prove that the scheme is symmetric and positive definite in general cases. In §4, we point out that the stencil of the scheme is equal or less than a nine point stencil on two dimensional quadrangular meshes. Additionally, there is a relationship with the formula used to compute edge unknowns between our scheme and [3]. In §5, we prove that the scheme is convergent for discontinuous tensors. In §6, it is devoted to numerical tests. We compare these numerical results between our scheme and some other methods in the benchmark FVCA 5.

2 Notations

Let Ω be an open bounded polygonal set of $\mathbb{R}^2$ with the boundary $\partial\Omega$. We denote three discretization families of Ω by $\mathcal{D}$, $\mathcal{D}^*$ and $\mathcal{D}^{**}$, which are given by
1. $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ such that
1a. $\mathcal{M}$ is a finite family of non empty connected open disjoint subsets of Ω such that $\overline{\Omega} = \bigcup\limits_{K\in\mathcal{M}} \overline{K}$. $m_K > 0$ denotes the measure of K (the "primary control volume").
1b. $\mathcal{E}$ (the set of edges of the primary grid) is a set of disjoint subsets of $\overline{\Omega}$ such that, for all $\sigma \in \mathcal{E}$, σ is a segment in $\mathbb{R}$. We denote by m_σ the measure of σ. Let

K be an element of $\mathcal{M}$, we assume that there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \sigma$ and $\mathcal{E} = \bigcup_{K\in\mathcal{M}} \mathcal{E}_K$. The set of interior edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$) with $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E} | \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E} | \sigma \subset \partial\Omega\}$).

1c. $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ is a set of mesh points of the primary grid. For all $K \in \mathcal{M}$, $x_K \in K$ and K is assumed to be x_K-star-shaped, i.e for all $x \in K$, $[x_K, x] \in K$.



A sample primary mesh (solid lines) and its dual mesh (dashed lines)

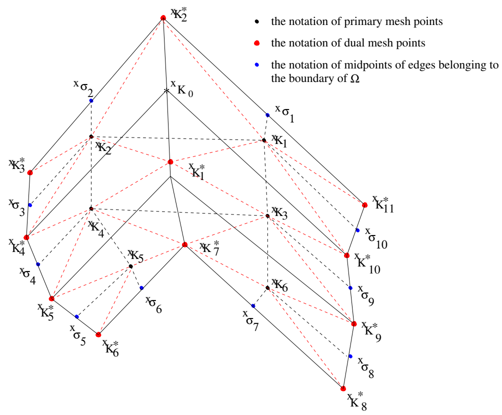
2. $\mathcal{D}^* = (\mathcal{M}^*, \mathcal{E}^*, \mathcal{P}^*, \mathcal{V}^*)$ such that
2a. The dual control volumes K^* are defined by connecting mesh points of the primary control volumes and the midpoints of the edges belonging to $\partial\Omega$. Moreover, we assume that the lines joining their mesh points are inside Ω. In this case, $\mathcal{M}^*$ which is a set of dual control volumes such that $\overline{\Omega} = \bigcup_{K^*\in\mathcal{M}^*} \overline{K^*}$, is defined and we assume it fits the initial domain Ω. We denote by $m_{K^*} > 0$ the measure of K^*.

Remark 2.1: A sufficient condition to define the mesh $\mathcal{M}^*$ is that, for neighboring control volumes, the line joining their centers intersects their common edge. This condition is not necessary.

2b. $\mathcal{E}^*$ (the set of edges of the dual grid) is a set of disjoint subsets of $\overline{\Omega}$ such that, for all $\sigma^* \in \mathcal{E}^*$, σ^* is a segment in $\mathbb{R}$. We denote by m_{σ^*} the measure of σ^*. Let K^* be an element of $\mathcal{M}^*$, we assume that there exists a subset $\mathcal{E}_{K^*}$ of $\mathcal{E}^*$ such that $\partial K^* = \bigcup_{\sigma\in\mathcal{E}_{K^*}} \sigma$ and $\mathcal{E}^* = \bigcup_{K^*\in\mathcal{M}^*} \mathcal{E}_{K^*}$.
2c. $\mathcal{P}^* = (x_{K^*})_{K^*\in\mathcal{M}^*}$ is the set of mesh points of the dual grid.

2d. $\mathcal{V}^*$ is the set of vertices of the dual meshes which includes the primary mesh points, the midpoints of the edges belonging to $\partial\Omega$ and the boundary vertices of Ω.

Remark 2.2: We do not always use the vertices of the primary mesh as dual mesh points $\{x_{K^*}\}_{K^*\in\mathcal{M}^*}$. For example, we consider the following polygon Ω:



we can not choose the vertex x_{K_0} of the primary mesh to define the dual mesh point of $(x_{K_1}, x_{K_2}, x_{K_3}, x_{K_4}) \in \mathcal{M}^*$, because it is outside $(x_{K_1}, x_{K_2}, x_{K_3}, x_{K_4})$.

3. $\mathcal{D}^{**} = (\mathcal{M}^{**}, \mathcal{V}^{**})$
Now, we construct a third grid.
Let be $K^* \in \mathcal{M}^*$, if all the edges of $\mathcal{E}_{K^*}$ do not belong to the boundary $\partial\Omega$, the set of vertices of K^* only contains mesh points of the primary control volumes. A point x_{K^*} is chosen inside K^* and connected to all vertices of K^*. We have the two following examples to describe the construction of the third mesh:

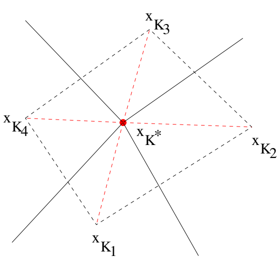


Figure 2.1

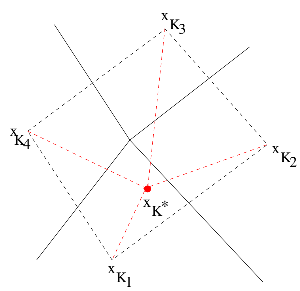


Figure 2.2

If K^* has a vertex x_{K^*} belonging to the boundary of Ω, its dual mesh point is equal to the vertex x_{K^*} (see figure 2.4). We connect x_{K^*} to the other vertices of K^*.

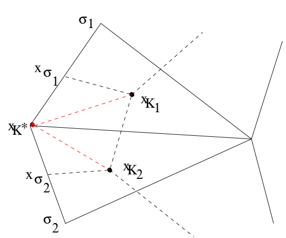


Figure 2.4

In the above figures, we denote that:

- The primary mesh is represented by solid black lines.
- The dual mesh is represented by dashed black lines.

- The third mesh is represented by dashed red and black lines.
- The primary mesh points $x_{K_1}, x_{K_2}, x_{K_3}, x_{K_4}$ are elements of $\mathcal{P}$.
- The dual mesh points x_{K^*}, x_{L^*} are elements of $\mathcal{P}^*$.
- The edges σ_1, σ_2 are edges of the boundary of Ω.
- The points x_{σ_1}, x_{σ_2} are midpoints of the edges σ_1, σ_2.

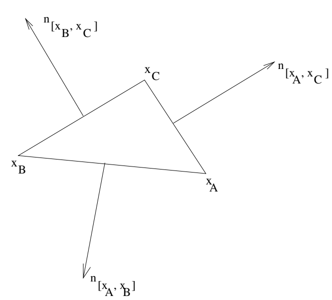
From the construction of the third grid, this implies that it is a subgrid of the dual grid.
3a. $\mathcal{M}^{**}$ is a finite family of sub-triangles such that $\overline{\Omega} = \underset{T\in\mathcal{M}^{**}}{\cup} \overline{T}$.

3b. $\mathcal{V}^{**}$ is a finite set of vertices of the third grid such that, for all $T \in \mathcal{M}^{**}, \mathcal{V}^{**}_T$ is a set of three vertices of triangle T and $\mathcal{V}^{**} = \bigcup\limits_{T\in\mathcal{M}^{**}} \mathcal{V}^{**}_T$. The set of interior vertices is denoted by $\mathcal{V}^{**}_{\text{int}} = \mathcal{P} \cup \mathcal{P}^*$. Moreover, the functions p_K and p_{K^*} with $K \in \mathcal{M}$, $K^* \in \mathcal{M}^*$ are piecewise linear continuous functions defined by

$$p_K(x) = \begin{cases} 1 & \text{at } x = x_K,\ x_K \in \mathcal{P}, \\ 0 & \text{at } x \in \mathcal{V}^{**}_{\text{int}} \setminus \{x_K\}, \\ 0 & \text{on } \partial\Omega. \end{cases}$$

$$p_{K^*}(x) = \begin{cases} 1 & \text{at } x = x_{K^*},\ x_{K^*} \in \mathcal{P}^*, \\ 0 & \text{at } x \in \mathcal{V}^{**}_{\text{int}} \setminus \{x_{K^*}\}, \\ 0 & \text{on } \partial\Omega. \end{cases}$$

Additionally, we introduce some notations $n_{[x_A,x_B]}$, $n_{[x_A,x_C]}$, $n_{[x_B,x_C]}$ which are outward normal vectors of the triangle (x_A, x_B, x_C). The lengths of these vectors are equal to the segments $[x_A, x_B]$, $[x_A, x_C]$, $[x_B, x_C]$ and $m_{(x_A,x_B,x_C)}$ is the measure of the triangle (x_A, x_B, x_C).



3 Presentation of the scheme

Now, we introduce our scheme: *A Cell-Centered Scheme For Heterogeneous Anisotropic Diffusion Problems On General Meshes.* We name it FECC for the Finite Element Cell-Centered scheme.

Definition 3.1: Let us define the discrete function space $\mathcal{H}_\mathcal{D}$ as the set of all

$((u_K)_{K\in\mathcal{M}}, (u_{K^*})_{K^*\in\mathcal{M}^*})$, $u_K \in \mathbb{R}$, $K \in \mathcal{M}$, $u_{K^*} \in \mathbb{R}$, $K^* \in \mathcal{M}^*$ and $u_{K^*} = 0$ if x_{K^*} belongs to the boundary of Ω.

$P(v)$ which is a function on Ω, is constructed from the value $v = ((v_K)_{K\in\mathcal{M}}, (v_{K^*})_{K^*\in\mathcal{M}^*})$. The function $\nabla_{\mathcal{D},\Lambda} u$ is intended to be a discrete gradient of ∇u taking into account $u = ((u_K)_{K\in\mathcal{M}}, (u_{K^*})_{K^*\in\mathcal{M}^*})$. As a result, equation (2) is discretized by the following discrete variational formulation

$$\int_\Omega \left(\Lambda(x)\nabla_{\mathcal{D},\Lambda} u(x)\right).\nabla_{\mathcal{D},\Lambda} v(x)\, dx = \int_\Omega f(x)P(v)(x)dx \quad \text{for all } v \in \mathcal{H}_\mathcal{D}. \tag{3}$$

From equation (3), we describe the FECC scheme in each of the following cases of Λ.

3.1 Isotropic homogeneous case: $\Lambda = Id$

The main idea of the FECC scheme is the same as that of the standard finite element method ($P1$) on the third triangular mesh. The domain Ω is partitioned by this third mesh.
For any $u = ((u_K)_{K\in\mathcal{M}}, (u_{K^*})_{K^*\in\mathcal{M}^*}) \in \mathcal{H}_\mathcal{D}$, $P(u)$ is defined by

$$P(u)(x) = \sum_{K\in\mathcal{M}} u_K.p_K(x) + \sum_{K^*\in\mathcal{M}^*} u_{K^*}.p_{K^*}(x), \tag{4}$$

and $\nabla_{\mathcal{D},\Lambda} u$ is defined by

$$\nabla_{\mathcal{D},Id} u(x) = \nabla_{\mathcal{D},Id} P(u)(x) = \sum_{K\in\mathcal{M}} u_K \nabla p_K(x) + \sum_{K^*\in\mathcal{M}^*} u_{K^*}\nabla p_{K^*}(x). \tag{5}$$

Substituting definitions (4) and (5) into equation (3), for each $L \in \mathcal{M}\cup\mathcal{M}^*$, we choose $v = ((v_K)_{K\in\mathcal{M}}, (v_{K^*})_{K^*\in\mathcal{M}^*}) \in \mathcal{H}_\mathcal{D}$ such that $v_K = 0$ if $K \neq L$, $v_{K^*} = 0$ if $K^* \neq L$, $v_L = 1$ and $P(v) = p_L$. The resulting equation can be re-written in the following form:

$$\int_\Omega \left(\sum_{K\in\mathcal{M}} u_K \nabla p_K + \sum_{K^*\in\mathcal{M}^*} u_{K^*}\nabla p_{K^*}\right).\nabla p_L\; dx = \int_\Omega f.p_L\; dx. \tag{6}$$

We present the construction of the FECC scheme in an isotropic homogeneous case.
Step 1 *Recover all u_{K^*} with $K^* \in \mathcal{M}^*$ by linear functions of $(u_K)_{K\in\mathcal{M}}$ and constants depending on function f.*
We choose p_L equal to p_{K^*}. We have

$$\int_\Omega \left(\left(\sum_{K\in\mathcal{M}} u_K\nabla p_K\right).\nabla p_{K^*} + u_{K^*}\nabla p_{K^*}.\nabla p_{K^*}\right)\; dx = \int_\Omega f.p_{K^*}\; dx,$$

because $supp\{p_{K^*}\}$ is a subset of K^* for all $K^* \in \mathcal{M}^*$.
Thus, u_{K^*} is equal to $\Pi_{K^*}\left(\{u_K\}_{K\in\mathcal{M}}, f\right)$ defined by

$$\Pi_{K^*}\left(\{u_K\}_{K\in\mathcal{M}}, f\right) = -\sum_{K\in\mathcal{M}} u_K \int_\Omega \nabla p_K(x)\frac{\nabla p_{K^*}(x)}{\|\nabla p_{K^*}\|^2_{L^2(\Omega)}}\, dx + \int_\Omega f(x)\frac{p_{K^*}(x)}{\|\nabla p_{K^*}\|^2_{L^2(\Omega)}}\, dx.$$

Step 2 *Transform the variables in formula (4).*

$$P(u)(x) = \sum_{K \in \mathcal{M}} u_K . p_K(x) + \sum_{K^* \in \mathcal{M}^*} \Pi_{K^*} \left(\{u_K\}_{K \in \mathcal{M}}, f \right) . p_{K^*}(x). \tag{7}$$

Step 3 *Construct a system of linear equations.*
Substituting (7) into (6), for each p_L belonging to $\{p_K\}_{K \in \mathcal{M}}$, we get:

$$\int_\Omega \left(\sum_{K \in \mathcal{M}} u_K \left(\nabla p_K . \nabla p_L \right) + \sum_{K^* \in \mathcal{M}^*} \Pi_{K^*} \left(\{u_K\}_{K \in \mathcal{P}}, f \right) \left(\nabla p_{K^*} . \nabla p_L \right) \right) dx = \int_\Omega f . p_L \, dx. \tag{8}$$

This is a linear equation which only involves the cell unknowns $\{u_K\}_{K \in \mathcal{M}}$. Hence, we construct a system of linear equations

$$A.U = B, \tag{9}$$

where U is a vector $(u_K)_{K \in \mathcal{M}}$ and A is a square matrix in $\mathbb{R}^{card(\mathcal{M}) \times card(\mathcal{M})}$. All unknowns $(u_{K^*})_{K^* \in \mathcal{P}^*}$ have been eliminated, the scheme is thus indeed cell-centered.

Lemma 3.1**:** In isotropic homogeneous cases, the matrix A of system (9) is symmetric and positive definite on general meshes. Moreover, the scheme is convergent.

Proof of lemma 3.1**:**
The matrix A is proved to be symmetric and positive definite as lemma 3.2, and the scheme is equivalent to the standard finite element method on the third grid when we consider the isotropic homogeneous case. Therefore, it is convergent (see details of the proof in 2.4 and chapter 3 of [9]).

3.2 Anisotropic heterogeneous case

To simplify the description of the FECC scheme, we assume that, for neighboring control volumes, the line joining their primary mesh points intersects their common edge. Hence, the dual mesh is centered around the vertices of the primary mesh and the dual mesh points are the vertices of the primary mesh. Taking any $\sigma \in \mathcal{E}_{\text{int}}$ such that $\mathcal{M}_\sigma = \{K, L\}$, $x_K, x_L \in \mathcal{P}$ and $x_{K^*} \in \mathcal{P}^*$, we denote by (x_K, x_L, x_{K^*}) a triangle of $\mathcal{M}^{**}$. We take the values $u_{K^*},\ u_K,\ u_L$ of u at $x_{K^*},\ x_K,\ x_L$.

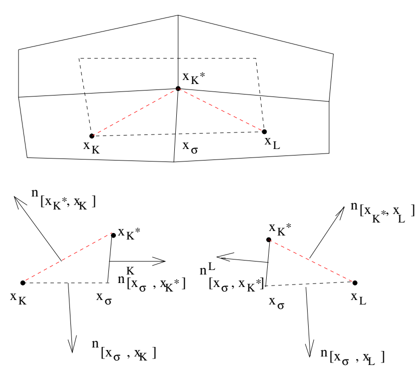


Figure 3.1

From values u_{K^*}, u_K, u_L, we want to construct a discrete gradient $\nabla_{\mathcal{D},\Lambda} u$ on the triangle (x_{K^*}, x_K, x_L) taking into account the heterogeneity of Λ. We consider the function

$$P_{(K^*,K,L)}(u) \;:\quad (x_{K^*}, x_K, x_L) \to \mathbb{R},$$

where it is continuous, linear on (x_{K^*}, x_K, x_σ) and (x_{K^*}, x_L, x_σ) (two half triangles of (x_K, x_L, x_{K^*})). We introduce a value $u_\sigma^{K^*}$ (a temporary unknown) at x_σ. The point x_σ is an intersecting point between the line joining two mesh points x_K, x_L and the internal edge σ. The discrete gradient $\nabla_{\mathcal{D},\Lambda} u$ is then defined by

- on the triangle (x_{K^*}, x_K, x_σ)

$$P_{(K^*,K,L)}(u)(x) = \begin{cases} u_K & x = x_K, \\ u_{K^*} & x = x_{K^*}, \\ u_\sigma^{K^*} & x = x_\sigma. \end{cases}$$

$$\begin{aligned} &\nabla_{\mathcal{D},\Lambda} u = \nabla_{\mathcal{D},\Lambda} P_{(K^*,K,L)}(u) \\ = \;& \frac{-P_{(K^*,K,L)}(u)(x_\sigma) n_{[x_{K^*},x_K]} - P_{(K^*,K,L)}(u)(x_K) n^K_{[x_\sigma,x_{K^*}]} - P_{(K^*,K,L)}(u)(x_{K^*}) n_{[x_\sigma,x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}} \\ = \;& \frac{-u_\sigma^{K^*} n_{[x_{K^*},x_K]} - u_K n^K_{[x_\sigma,x_{K^*}]} - u_{K^*} n_{[x_\sigma,x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}}, \end{aligned}$$

  where $n^K_{[x_\sigma,x_{K^*}]}$ is the outer normal vector to the triangle (x_{K^*}, x_K, x_σ). The length of the vector $n^K_{[x_\sigma,x_{K^*}]}$ is equal to the length of the segment $[x_\sigma, x_{K^*}]$. If x_σ belongs to the boundary of Ω then $u_\sigma^{K^*} = 0$.

- on the triangle (x_{K^*}, x_L, x_σ)

$$P_{(K^*,K,L)}(u)(x) = \begin{cases} u_L & x = x_K, \\ u_{K^*} & x = x_{K^*}, \\ u_\sigma^{K^*} & x = x_\sigma. \end{cases}$$

$$\begin{aligned} &\nabla_{\mathcal{D},\Lambda} u = \nabla_{\mathcal{D},\Lambda} P_{(K^*,K,L)}(u) \\ = \;& \frac{-P_{(K^*,K,L)}(u)(x_\sigma) n_{[x_{K^*},x_L]} - P_{(K^*,K,L)}(u)(x_L) n^L_{[x_\sigma,x_{K^*}]} - P_{(K^*,K,L)}(u)(x_{K^*}) n_{[x_\sigma,x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \\ = \;& \frac{-u_\sigma^{K^*} n_{[x_{K^*},x_L]} - u_L n^L_{[x_\sigma,x_{K^*}]} - u_{K^*} n_{[x_\sigma,x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}}, \end{aligned}$$

  where $n^L_{[x_\sigma,x_{K^*}]}$ is the outer normal vector to the triangle (x_{K^*}, x_L, x_σ). The length of the vector $n^L_{[x_\sigma,x_{K^*}]}$ is equal to the length of segment $[x_\sigma, x_{K^*}]$. If x_σ belongs to boundary of Ω then $u_\sigma^{K^*} = 0$.

These definitions depend on $u_\sigma^{K^*}$ but we fix $u_\sigma^{K^*}$ by imposing the Local Conservativity of the Fluxes, i.e

$$\Lambda_K \left(\nabla_{\mathcal{D},\Lambda} u\right)_{|(x_{K^*},x_K,x_\sigma)} .n^K_{[x_\sigma,x_{K^*}]} + \Lambda_L \left(\nabla_{D,\Lambda} u\right)_{|(x_{K^*},x_L,x_\sigma)} .n^L_{[x_\sigma,x_{K^*}]} = 0, \qquad (10)$$

where Λ_K, Λ_L are values of Λ on K and L.
Equation (10) corresponds to the following equation:

$$u_\sigma^{K^*} = \beta_K^{K^*,\sigma} u_K + \beta_L^{K^*,\sigma} u_L + \beta_{K^*}^{K^*,\sigma} u_{K^*}, \tag{11}$$

with

$$\beta_K^{K^*,\sigma} = \left(\frac{(n^K_{[x_\sigma,x_{K^*}]})^T \Lambda_K n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}} \right) \Big/ \left(-\frac{(n^K_{[x_\sigma,x_{K^*}]})^T \Lambda_K n_{[x_{K^*},x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}} - \frac{(n^L_{[x_\sigma,x_{K^*}]})^T \Lambda_L n_{[x_{K^*},x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \right),$$

$$\beta_L^{K^*,\sigma} = \left(\frac{(n^L_{[x_\sigma,x_{K^*}]})^T \Lambda_L n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \right) \Big/ \left(-\frac{(n^K_{[x_\sigma,x_{K^*}]})^T \Lambda_K n_{[x_{K^*},x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}} - \frac{(n^L_{[x_\sigma,x_{K^*}]})^T \Lambda_L n_{[x_{K^*},x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \right),$$

$$\beta_{K^*}^{K^*,\sigma} = 1 - \beta_K^{K^*,\sigma} - \beta_L^{K^*,\sigma}.$$

From equation (11), the unknown $u_\sigma^{K^*}$ is computed using u_K, u_{K^*} and u_L. Thus, the discrete gradient $\nabla_{\mathcal{D},\Lambda} u$ on (x_K, x_L, x_{K^*}) only depends on these three values.

Hypothesis 3.1: we assume

$$\left(-\frac{(n^K_{[x_\sigma,x_{K^*}]})^T \Lambda_K n_{[x_{K^*},x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}} - \frac{(n^L_{[x_\sigma,x_{K^*}]})^T \Lambda_L n_{[x_{K^*},x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \right) \neq 0.$$

Remark 3.2:

- In isotropic heterogeneous cases, if the primary mesh is an admissible mesh (see definition 3.1, paper $37-39$ in [17]), the unknown $u_\sigma^{K^*}$ is computed by

$$u_\sigma^{K^*} = \underbrace{\beta_K^{K^*,\sigma}}_{>0} u_K + \underbrace{\beta_L^{K^*,\sigma}}_{>0} u_L,$$

  because $n^K_{[x_\sigma,x_{K^*}]}.\ n_{[x_\sigma,x_K]} = 0$ and $n^L_{[x_\sigma,x_{K^*}]}.\ n_{[x_\sigma,x_L]} = 0$.

- In isotropic homogeneous cases, we do not need hypothesis 3.1, since the coefficients are different from 0.

We present the construction of the FECC scheme in anisotropic heterogeneous cases.
Step 1 *Recover all u_{K^*} with $K^* \in \mathcal{M}^*$ by linear functions of $(u_K)_{K\in\mathcal{M}}$ and constants depending on function f.*
For each $K^* \in \mathcal{M}^*$, we choose $v = (\{v_L\}_{L\in\mathcal{M}}, \{v_{L^*}\}_{L^*\in\mathcal{M}^*})$ such that $v_L = 0$ for all $L \in \mathcal{M}$, $v_{L^*} = 0$ if $L^* \neq K^*$ and $v_{K^*} = 1$ in equation (3)

$$\int_\Omega \left(\Lambda(x)\nabla_{\mathcal{D},\Lambda} u(x)\right).\nabla_{\mathcal{D},\Lambda} v(x)\, dx = \int_\Omega f(x)P(v)(x)dx.$$

The discrete gradient $\nabla_{\mathcal{D},\Lambda} v$ is equal to 0 on L^* which is different from K^*. It implies that $\int_\Omega \left(\Lambda(x)\nabla_{\mathcal{D},\Lambda} u(x)\right).\nabla_{\mathcal{D},\Lambda} v(x)\, dx$ is presented by the linear function of $(u_K)_{K\in\mathcal{M}}$, u_{K^*} and a constant depending on function f. Therefore, the unknown

u_{K^*} is computed by a linear function of $\{u_K\}_{K\in\mathcal{M}}$ and a constant depending on function f. This linear function is also denoted by $\Pi_{K^*}\left(\{u_K\}_{K\in\mathcal{M}}, f\right)$.

Step 2 *Reconstruct the discrete gradient* $\nabla_{\mathcal{D},\Lambda}u$.
In the definition of the discrete gradient $\nabla_{\mathcal{D},\Lambda}u$, we transform all the unknowns $\{u_{K^*}\}_{K^*\in\mathcal{M}^*}$ by $\left\{\Pi_{K^*}\left(\{u_K\}_{K\in\mathcal{M}}, f\right)\right\}_{K^*\in\mathcal{M}^*}$. Hence, $\nabla_{\mathcal{D},\Lambda}u$ does not depend on unknowns $\{u_{K^*}\}_{K^*\in\mathcal{M}^*}$.

Step 3 *Construct a system of linear equations.*
In equation (3), for each $K \in \mathcal{M}$, we choose $v = (\{v_L\}_{L\in\mathcal{M}}, \{v_{L^*}\}_{L^*\in\mathcal{M}^*}) \in \mathcal{H}_{\mathcal{D}}$ such that $v_{L^*} = 0$ for all $L^* \in \mathcal{M}^*$, $v_L = 0$ if $L \neq K$ and $v_K = 1$. This resulting equation is a linear equation which only involves unknowns $\{u_K\}_{K\in\mathcal{M}}$. Thus, we construct a system of linear equations

$$A.U = B, \tag{12}$$

where U is the vector $(u_K)_{K\in\mathcal{M}}$ and A is a square matrix in $\mathbb{R}^{card(\mathcal{M})\times card(\mathcal{M})}$.

Lemma 3.2**:** With hypothesis (3.1), for anisotropic heterogeneous cases, the matrix A of system (12) is symmetric and positive definite on general meshes.

Proof of lemma 3.2**:**
By definition, the discrete gradient $\nabla_{\mathcal{D},\Lambda}u$ depends on elements of sets $\{u_K\}_{K\in\mathcal{M}}$, $\{u_{K^*}\}_{K^*\in\mathcal{M}^*}$ and $\left\{u_\sigma^{K^*}\right\}_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}}$. The set $\left\{u_\sigma^{K^*}\right\}_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}}$ is only considered in anisotropic heterogeneous cases. Hence, we can present

$$\int_\Omega (\Lambda\nabla_{\mathcal{D},\Lambda}u)\,.\nabla_{\mathcal{D},\Lambda}v dx = \mathcal{U}^T A_\Lambda \mathcal{V},$$

where $\mathcal{U}$, $\mathcal{V}$ are defined by

$$\mathcal{U} = \begin{pmatrix} (u_{K^*})_{K^*\in\mathcal{M}^*} \\ (u_K)_{K\in\mathcal{M}} \\ \left(u_\sigma^{K^*}\right)_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}} \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} (v_{K^*})_{K^*\in\mathcal{M}^*} \\ (v_K)_{K\in\mathcal{M}} \\ \left(v_\sigma^{K^*}\right)_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}} \end{pmatrix},$$

$$\mathcal{M}_\sigma^* = \{K^* \in \mathcal{M}^* \text{ such that } \sigma\cap K^* \neq \emptyset\},$$

with

$$\begin{array}{rcl} (u_K)_{K\in\mathcal{M}},\ (v_K)_{K\in\mathcal{M}} & \in & M^{card(\mathcal{M})\times 1}, \\ (u_{K^*})_{K^*\in\mathcal{M}^*},\ (v_{K^*})_{K^*\in\mathcal{M}^*} & \in & M^{card(\mathcal{M}^*)\times 1}, \\ \left(u_\sigma^{K^*}\right)_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}},\ \left(v_\sigma^{K^*}\right)_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ K^*\in\mathcal{M}_\sigma^*}} & \in & M^{\left\{\sum\limits_{\sigma\in\mathcal{E}_{\text{int}}} card(\mathcal{M}_\sigma^*)\right\}\times 1}. \end{array}$$

Defining $m = card(\mathcal{M}) + card(\mathcal{M}^*) + \sum\limits_{\sigma\in\mathcal{E}_{\text{int}}} card(\mathcal{M}^*_\sigma)$ and $n = card(\mathcal{M}) + card(\mathcal{M}^*)$, we obtain

$$A_\Lambda = \left((a_\Lambda)_{ij}\right)_{i,j\in\overline{1,m}} \quad \text{and} \quad \mathcal{U},\mathcal{V} \in M^{m\times 1}.$$

Moreover, A_Λ is symmetric. This is deduced from $\int\limits_\Omega \left(\Lambda\nabla_{D,\Lambda}u\right).\nabla_{D,\Lambda}v dx = \int\limits_\Omega \nabla_{D,\Lambda}u.\left(\Lambda\nabla_{D,\Lambda}v\right)dx$, because the tensor Λ is symmetric.
In anisotropic heterogeneous cases, the discrete gradient $\nabla_{\mathcal{D},\Lambda}u$ can be re-written by

- on the triangle (x_{K^*}, x_K, x_σ)

$$\nabla_{D,\Lambda}u = -\frac{\left[\begin{array}{l}\left(\beta_K^{K^*,\sigma} n_{[x_{K^*},x_K]} + n^K_{[x_\sigma,x_{K^*}]}\right)u_K + \left(\beta_L^{K^*,\sigma} n_{[x_{K^*},x_K]}\right)u_L \\ +\left(\beta_{K^*}^{K^*,\sigma} n_{[x_{K^*},x_K]} + n_{[x_\sigma,x_K]}\right)u_{K^*}\end{array}\right]}{2m_{(x_{K^*},x_K,x_\sigma)}}, \tag{13}$$

- on the triangle (x_{K^*}, x_L, x_σ)

$$\nabla_{D,\Lambda}u = -\frac{\left[\begin{array}{l}\left(\beta_K^{K^*,\sigma} n_{[x_{K^*},x_L]}\right)u_K + \left(\beta_L^{K^*,\sigma} n_{[x_{K^*},x_L]} + n^L_{[x_\sigma,x_{K^*}]}\right)u_L \\ +\left(\beta_{K^*}^{K^*,\sigma} n_{[x_{K^*},x_L]} + n_{[x_\sigma,x_L]}\right)u_{K^*}\end{array}\right]}{2m_{(x_{K^*},x_L,x_\sigma)}}, \tag{14}$$

because $u_\sigma^{K^*} = \beta_K^{K^*,\sigma}u_K + \beta_L^{K^*,\sigma}u_L + \beta_{K^*}^{K^*,\sigma}u_{K^*}$.

Now, the discrete gradient $\nabla_{\mathcal{D},\Lambda}u$ depends on elements of set $\left\{(u_K)_{K\in\mathcal{M}}, (u_{K^*})_{K^*\in\mathcal{M}^*}\right\}$ in general cases. Therefore, there exists a matrix $C^* \in M^{m\times n}$ such that $\mathcal{U} = C^*\mathcal{U}^*$ with $\mathcal{U} \neq 0$, which implies

$$\mathcal{U}^T A_\Lambda \mathcal{U} = (\mathcal{U}^*)^T \left(C^{*T} A_\Lambda C^*\right)\mathcal{U}^*,$$

where $C^* \in M^{m\times n}$ and $\mathcal{U}^* = \begin{pmatrix} (u_{K^*})_{K^*\in\mathcal{M}^*} \\ (u_K)_{K\in\mathcal{M}} \end{pmatrix} \in M^{n\times 1}$. As the matrix A_Λ is symmetric, the matrix $G = C^{*T}A_\Lambda C^*$ is also symmetric.

According to step 1 of the construction scheme, for each $K^* \in \mathcal{M}^*$, we choose $v = (\{v_L\}_{L\in\mathcal{M}}, \{v_{L^*}\}_{L^*\in\mathcal{M}^*})$ such that $v_L = 0$ for all $L \in \mathcal{M}$, $v_{L^*} = 0$ if $L^* \neq K^*$ and $v_{K^*} = 1$ in equation (3)

$$\int\limits_\Omega \left(\Lambda(x)\nabla_{\mathcal{D},\Lambda}u(x)\right).\nabla_{\mathcal{D},\Lambda}v(x)\,dx = \int\limits_\Omega f(x)P(v)(x)dx.$$

From these linear equations, the first system of linear equations is constructed by

$$DU^* + EU = F^*, \tag{15}$$

where $F^* \in M^{card(\mathcal{M}^*)\times 1}$, $D \in M^{card(\mathcal{M}^*)\times card(\mathcal{M}^*)}$ and $E \in M^{card(\mathcal{M}^*)\times card(\mathcal{M})}$.

According to step 3 of the construction scheme, for each $K \in \mathcal{M}$, we choose $v = (\{v_L\}_{L\in\mathcal{M}}, \{v_{L^*}\}_{L^*\in\mathcal{M}^*}) \in \mathcal{H}_\mathcal{D}$ such that $v_{L^*} = 0$ for all $L^* \in \mathcal{M}^*$, $v_L = 0$ if $L \neq K$ and $v_K = 1$ in equation (3). We get the second system of linear equations, as follows:

$$MU^* + NU = F, \tag{16}$$

where $F \in M^{card(\mathcal{M})\times 1}$, $N \in M^{card(\mathcal{M})\times card(\mathcal{M})}$ and $M \in M^{card(\mathcal{M})\times card(\mathcal{M}^*)}$. Both F and F^* depend on function f.
From (15) and (16), it follows that $G = \begin{pmatrix} D & E \\ M & N \end{pmatrix}$ where $M = E^T$ and two square matrices D, N are symmetric, because the matrix G is symmetric.

Next, we prove that the matrix G is positive definite. Assume that $\mathcal{U}^* \neq 0$, there are two following cases:
In the first case where $u_K \neq 0$ for all $K \in \mathcal{M}$, we consider $T_0 = (x_{K^*}, x_{\sigma_1}, x_K) \in \mathcal{M}^{**}$. This triangle has an edge belonging to the boundary of Ω presented in the following figure:

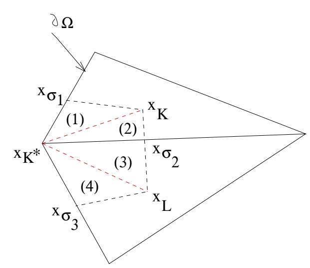


Figure 3.2

On the triangle $T_0 = (x_{K^*}, x_{\sigma_1}, x_K)$, the discrete gradient $\nabla_{D,\Lambda}u$ is defined by

$$\nabla_{D,\Lambda}u = \frac{-u_K n_{[x_{K^*},x_{\sigma_1}]}}{2m_{(x_{K^*},x_{\sigma_1},x_K)}},$$

because $u_{K^*} = u_{\sigma_1}^{K^*} = 0$.
All eigenvalues of tensor Λ are equal or greater than $\underline{\lambda} > 0$, thus

$$\begin{aligned} \mathcal{U}^{*T} G \mathcal{U}^* = \int_\Omega (\Lambda \nabla_{\mathcal{D},\Lambda}u) . \nabla_{\mathcal{D},\Lambda}u dx \quad &\geq \quad \underline{\lambda} \int_\Omega (\nabla_{\mathcal{D},\Lambda}u)^2 \, dx \\ &\geq \quad \underline{\lambda} \int_{T_0} (\nabla_{D,\Lambda}u)^2 \, dx = \underline{\lambda} \frac{\left(u_K n_{[x_{K^*},x_{\sigma_1}]}\right)^2}{4m_{(x_{K^*},x_{\sigma_1},x_K)}} > 0. \end{aligned}$$

In the second case, there exists $K \in \mathcal{M}$ such that $u_K = 0$. In this case, we have a triangle $T_0 = (x_{K^*}, x_K, x_L)$ such that $u_L \neq 0$ or $u_{K^*} \neq 0$ (see figure 3.1).
The integral $\int_{T_0} (\nabla_{D,\Lambda}u)^2 \, dx$ is computed by

$$\int_{T_0} (\nabla_{D,\Lambda}u)^2 \, dx = \int_{(x_{K^*},x_\sigma,x_K)} (\nabla_{D,\Lambda}u)^2 \, dx + \int_{(x_{K^*},x_\sigma,x_L)} (\nabla_{D,\Lambda}u)^2 \, dx.$$

On the triangle (x_{K^*}, x_σ, x_K), the discrete gradient $\nabla_{\mathcal{D},\Lambda} u$ is defined by

$$\nabla_{D,\Lambda} u = \frac{-u_{K^*}\left(n_{[x_K,x_\sigma]} + \beta_{K^*}^{K^*,\sigma} n_{[x_K,x_{K^*}]}\right) - u_L \beta_L^{K^*,\sigma} n_{[x_K,x_{K^*}]}}{2m_{(x_{K^*},x_\sigma,x_K)}}.$$

It follows that

$$\int\limits_{(x_{K^*},x_\sigma,x_K)} (\nabla_{D,\Lambda} u)^2 \, dx = \frac{\left\{u_{K^*}\left(n_{[x_K,x_\sigma]} + \beta_{K^*}^{K^*,\sigma} n_{[x_K,x_{K^*}]}\right) + u_L \beta_L^{K^*,\sigma} n_{[x_K,x_{K^*}]}\right\}^2}{4m_{(x_{K^*},x_\sigma,x_K)}} > 0,$$

since the direction of the vector $\left(n_{[x_K,x_\sigma]} + \beta_{K^*}^{K^*,\sigma} n_{[x_K,x_{K^*}]}\right)$ is different from direction of the vector $n_{[x_K,x_{K^*}]}$, $\beta_L^{K^*,\sigma} \neq 0$ (use hypothesis 3.1) and $\left[\begin{array}{l} u_L \neq 0, \\ u_{K^*} \neq 0. \end{array}\right.$
Similarly to the first case, we get

$$\mathcal{U}^{*T} G \mathcal{U}^* = \int\limits_\Omega (\Lambda \nabla_{\mathcal{D},\Lambda} u) . \nabla_{\mathcal{D},\Lambda} u dx \geq \underline{\lambda} \int\limits_\Omega (\nabla_{\mathcal{D},\Lambda} u)^2 \, dx \geq \underline{\lambda} \int\limits_{T_0} (\nabla_{D,\Lambda} u)^2 \, dx > 0.$$

Therefore, the matrix G is positive definite.

From (15), $\mathcal{U}^*$ is computed by $\mathcal{U}^* = D^{-1}(F^* - EU)$. In this formula, the matrix D^{-1} exists, because we apply the property of Schur complement of D in G (see more theorem 1.20, paper 44 in [33]) and G is symmetric, positive definite. Thus, (16) is transformed as follows:

$$\underbrace{\left(N - E^T D^{-1} E\right)}_{=A} U = \underbrace{F - E^T D^{-1} F^*}_{=B},$$

where the matrices A, B are defined in the systems of linear equations (9) and (12). Since G, D are symmetric, positive definite and using the property of Schur complement in G (see more theorem 1.12, paper 34 in [33]), we conclude that A is symmetric, positive definite. This allows us to use efficient methods to solve the systems of linear equations (9) and (12). □

4 Properties of the scheme

4.1 Isotropic homogeneous case

Property 4.1.1**:** The stencil of the FECC scheme is equal to 9 on quadrangular meshes.
Proof of property 4.1.1**:**
If the dual grid and the third grid are described by figure 4.1, then the stencil is equal to 9.

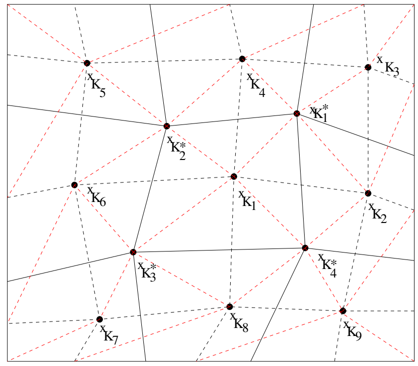


Figure 4.1

Step 1
The intersecting domains are not empty between $supp\{p_{K_1}\}$ and each of the following domains: $supp\{p_{K_2}\}$, $supp\{p_{K_4}\}$, $supp\{p_{K_6}\}$, $supp\{p_{K_8}\}$, $supp\{p_{K_1^*}\}$, $supp\{p_{K_2^*}\}$, $supp\{p_{K_3^*}\}$, $supp\{p^*_{K_4}\}$, and are empty between $supp\{p_{K_1}\}$ and the others: $supp\{p_{K_3}\}$, $supp\{p_{K_5}\}$, $supp\{p_{K_7}\}$, $supp\{p_{K_9}\}$. Therefore, equation (6) can be written as

$$\int_\Omega \begin{pmatrix} u_{K_1}\nabla p_{K_1} + u_{K_2}\nabla p_{K_2} + u_{K_4}\nabla p_{K_4} + u_{K_6}\nabla p_{K_6} + u_{K_8}\nabla p_{K_8} \\ +u_{K_1^*}\nabla p_{K_1^*} + u_{K_2^*}\nabla p_{K_2^*} + u_{K_3^*}\nabla p_{K_3^*} + u_{K_4^*}\nabla p_{K_4^*} \end{pmatrix} .\nabla p_{K_1} dx = \int_\Omega f.p_{K_1}\, dx.$$

Step 2

$$\begin{aligned}
u_{K_1^*} &= \alpha_1^{K_1^*} u_{K_1} + \alpha_2^{K_1^*} u_{K_2} + \alpha_3^{K_1^*} u_{K_3} + \alpha_4^{K_1^*} u_{K_4} + \alpha_{K_1^*}(f). \\
u_{K_2^*} &= \alpha_1^{K_2^*} u_{K_1} + \alpha_4^{K_2^*} u_{K_4} + \alpha_5^{K_2^*} u_{K_5} + \alpha_6^{K_2^*} u_{K_6} + \alpha_{K_2^*}(f). \\
u_{K_3^*} &= \alpha_1^{K_3^*} u_{K_1} + \alpha_6^{K_3^*} u_{K_6} + \alpha_7^{K_3^*} u_{K_7} + \alpha_8^{K_3^*} u_{K_8} + \alpha_{K_3^*}(f). \\
u_{K_4^*} &= \alpha_1^{K_4^*} u_{K_1} + \alpha_2^{K_4^*} u_{K_2} + \alpha_8^{K_4^*} u_{K_8} + \alpha_9^{K_4^*} u_{K_9} + \alpha_{K_4^*}(f).
\end{aligned}$$

Step 3

$$\int_\Omega \begin{pmatrix} u_{K_1}\nabla p_{K_1} + u_{K_2}\nabla p_{K_2} + u_{K_4}\nabla p_{K_4} + u_{K_6}\nabla p_{K_6} + u_{K_8}\nabla p_{K_8} \\ + \left(\alpha_1^{K_1^*} u_{K_1} + \alpha_2^{K_1^*} u_{K_2} + \alpha_3^{K_1^*} u_{K_3} + \alpha_4^{K_1^*} u_{K_4}\right)\nabla p_{K_1^*} \\ + \left(\alpha_1^{K_2^*} u_{K_1} + \alpha_4^{K_2^*} u_{K_4} + \alpha_5^{K_2^*} u_{K_5} + \alpha_6^{K_2^*} u_{K_6}\right)\nabla p_{K_2^*} \\ + \left(\alpha_1^{K_3^*} u_{K_1} + \alpha_6^{K_3^*} u_{K_6} + \alpha_7^{K_3^*} u_{K_7} + \alpha_8^{K_3^*} u_{K_8}\right)\nabla p_{K_3^*} \\ + \left(\alpha_1^{K_4^*} u_{K_1} + \alpha_2^{K_4^*} u_{K_2} + \alpha_8^{K_4^*} u_{K_8} + \alpha_9^{K_4^*} u_{K_9}\right)\nabla p_{K_4^*} \end{pmatrix} .\nabla p_{K_1} dx =$$

$$\int_\Omega \left\{ f.p_{K_1} - \left[\alpha_{K_1^*}(f)\nabla p_{K_1^*} - \alpha_{K_2^*}(f)\nabla p_{K_2^*} - \alpha_{K_3^*}(f)\nabla p_{K_3^*} - \alpha_{K_4^*}(f)\nabla p_{K_4^*}\right].\nabla p_{K_1} \right\} dx.$$

This equation only depends on nine cell unknowns, so the stencil is equal to nine. □

Remark 4.1: In some particular cases, the stencil can be 7 or even 5. We show the two following examples for these cases:
a) If the dual grid and the third grid are described by figure 4.2, then the stencil is equal to 7.

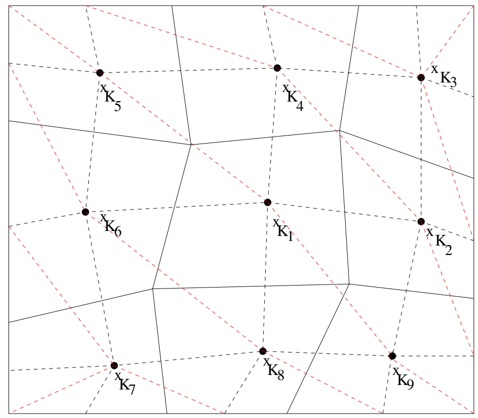


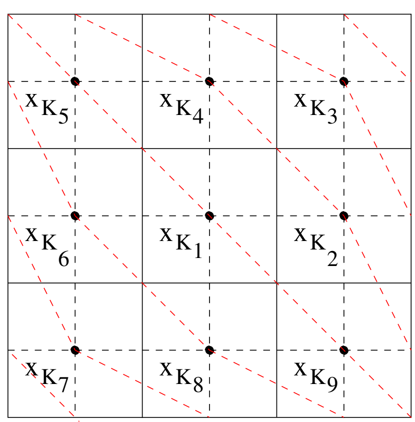
Figure 4.2

In figure 4.2, the polygon $(x_{K_2}, x_{K_4}, x_{K_5}, x_{K_6}, x_{K_8}, x_{K_9})$ is an element of the dual grid where the points x_{K^*} and x_{K_1} are the same.
The intersecting domains are not empty between $supp\{p_{K_1}\}$ and each of the following domains: $supp\{p_{K_2}\}$, $supp\{p_{K_4}\}$, $supp\{p_{K_5}\}$, $supp\{p_{K_6}\}$, $supp\{p_{K_8}\}$, $supp\{p_{K_9}\}$, and are empty between $supp\{p_{K_1}\}$ and the others: $supp\{p_{K_3}\}$, $supp\{p_{K_7}\}$. Hence, equation (6) can be written as:

$$\int_{\Omega} \left(\begin{array}{l} u_{K_1}\nabla p_{K_1} + u_{K_2}\nabla p_{K_2} + u_{K_4}\nabla p_{K_4} + u_{K_5}\nabla p_{K_5} \\ +u_{K_6}\nabla p_{K_6} + u_{K_8}\nabla p_{K_8} + u_{K_9}\nabla p_{K_9} \end{array} \right) .\nabla p_{K_1} dx = \int_{\Omega} f.p_{K_1}\, dx.$$

There are only seven main unknowns in the equation of step 1, thus the stencil is equal to seven.

b) We consider a particular case with a primary grid of squares. The dual grid is constructed on the following figure:



In the above figure, the polygon $(x_{K_2}, x_{K_4}, x_{K_5}, x_{K_6}, x_{K_8}, x_{K_9})$ is an element of the dual mesh where the point x_{K^*} and x_{K_1} are the same.
The intersecting domains are not empty between $supp\{p_{K_1}\}$ and each of the following domains: $supp\{p_{K_2}\}$, $supp\{p_{K_4}\}$, $supp\{p_{K_5}\}$, $supp\{p_{K_6}\}$, $supp\{p_{K_8}\}$, $supp\{p_{K_9}\}$, and are empty between $supp\{p_{K_1}\}$ and the others: $supp\{p_{K_3}\}$, $supp\{p_{K_7}\}$. Hence,

equation (6) can be written as:

$$\int\limits_{\Omega} \left(\begin{array}{l} u_{K_1}\nabla p_{K_1} + u_{K_2}\nabla p_{K_2} + u_{K_4}\nabla p_{K_4} + u_{K_5}\nabla p_{K_5} \\ +u_{K_6}\nabla p_{K_6} + u_{K_8}\nabla p_{K_8} + u_{K_9}\nabla p_{K_9} \end{array} \right).\nabla p_{K_1} dx = \int\limits_{\Omega} f.p_{K_1}\, dx.$$

Moreover, we have that:

- on the triangle $(x_{K_1}, x_{K_4}, x_{K_5})$
$$\nabla p_{K_5}.\ \nabla p_{K_1} = \frac{n_{[x_{K_1},x_{K_4}]}.\, n_{[x_{K_4},x_{K_5}]}}{\left(2m_{(x_{K_1},x_{K_4},x_{K_5})}\right)^2} = 0,$$
because $n_{[x_{K_1},x_{K_4}]}.\, n_{[x_{K_4},x_{K_5}]}$ is equal to 0,
- on the triangle $(x_{K_1}, x_{K_5}, x_{K_6})$
$$\nabla p_{K_5}.\ \nabla p_{K_1} = \frac{n_{[x_{K_1},x_{K_6}]}.\, n_{[x_{K_5},x_{K_6}]}}{\left(2m_{(x_{K_1},x_{K_5},x_{K_6})}\right)^2} = 0,$$
because $n_{[x_{K_1},x_{K_6}]}.\, n_{[x_{K_5},x_{K_6}]}$ is equal to 0,
- on the triangle $(x_{K_1}, x_{K_2}, x_{K_9})$
$$\nabla p_{K_9}.\nabla p_{K_1} = \frac{n_{[x_{K_1},x_{K_2}]}.\, n_{[x_{K_2},x_{K_9}]}}{\left(2m_{(x_{K_1},x_{K_2},x_{K_9})}\right)^2} = 0,$$
because $n_{[x_{K_1},x_{K_2}]}.\, n_{[x_{K_2},x_{K_9}]}$ is equal to 0 ,
- on the triangle $(x_{K_1}, x_{K_8}, x_{K_9})$
$$\nabla p_{K_9}.\nabla p_{K_1} = \frac{n_{[x_{K_1},x_{K_8}]}.\ n_{[x_{K_8},x_{K_9}]}}{\left(2m_{(x_{K_1},x_{K_8},x_{K_9})}\right)^2} = 0,$$
because $n_{[x_{K_1},x_{K_8}]}.\, n_{[x_{K_8},x_{K_9}]}$ is equal to 0 .

Hence, we get

$$\int\limits_{\Omega} \left(\begin{array}{l} u_{K_1}\nabla p_{K_1} + u_{K_2}\nabla p_{K_2} + u_{K_4}\nabla p_{K_4} \\ +u_{K_6}\nabla p_{K_6} + u_{K_8}\nabla p_{K_8} \end{array} \right).\nabla p_{K_1} dx = \int\limits_{\Omega} f.p_{K_1}\, dx,$$

which implies that the stencil is equal to 5. $\square$

We point out the next property of the FECC scheme.
According to the construction in isotropic cases, the affine function u on (x_{K^*}, x_K, x_L) is also affine on (x_{K^*}, x_K, x_σ) and (x_{K^*}, x_L, x_σ). In addition, this function has continuous fluxes because Λ is continuous. Therefore, it corresponds to the function u constructed in the heterogeneous cases, as follows:

$$\begin{aligned}
(\nabla_{D,Id}u)_{|(x_{K^*},x_K,x_\sigma)} &= \frac{-u_\sigma^{K^*} n_{[x_{K^*},x_K]} - u_K n^K_{[x_\sigma,x_{K^*}]} - u_{K^*} n_{[x_\sigma,x_K]}}{2m_{(x_{K^*},x_K,x_\sigma)}}, \\
(\nabla_{D,Id}u)_{|(x_{K^*},x_L,x_\sigma)} &= \frac{-u_\sigma^{K^*} n_{[x_{K^*},x_L]} - u_L n^L_{[x_\sigma,x_{K^*}]} - u_{K^*} n_{[x_\sigma,x_L]}}{2m_{(x_{K^*},x_L,x_\sigma)}},
\end{aligned}$$

where $u_\sigma^{K^*}$ is a temporary unknown. This unknown is eliminated by imposing the continuity of the fluxes:

$$(\nabla_{D,\Lambda} u)_{|(x_{K^*},x_K,x_\sigma)} .n^K_{[x_\sigma,x_{K^*}]} + (\nabla_{D,\Lambda} u)_{|(x_{K^*},x_L,x_\sigma)} .n^L_{[x_\sigma,x_{K^*}]} = 0.$$

$$\left(\alpha_\sigma^K + \alpha_\sigma^L\right) u_\sigma^{K^*} = \alpha_K u_K + \alpha_L u_L + \left(\alpha^K_{K^*} + \alpha^L_{K^*}\right) u_{K^*}, \tag{17}$$

where

$$\alpha_K = -\frac{n^K_{[x_\sigma,x_{K^*}]} .\, n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}}\ ,\ \alpha^K_{K^*} = -\frac{n_{[x_\sigma,x_K]} .\, n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}}\ ,\ \alpha^K_\sigma = \frac{n_{[x_{K^*},x_K]} .\, n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}},$$

$$\alpha_L = -\frac{n^L_{[x_\sigma,x_{K^*}]} .\, n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}}\ ,\ \alpha^L_{K^*} = -\frac{n_{[x_\sigma,x_L]} .\, n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}}\ ,\ \alpha^L_\sigma = \frac{n_{[x_{K^*},x_L]} .\, n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}}.$$

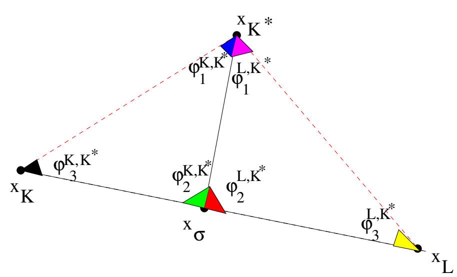
The following property presents a formula to compute the unknown u_σ in terms of the unknowns $u_K,\ u_L$.

Property 4.1.2: The unknown $u_\sigma^{K^*}$ of (17) satisfies

$$u_\sigma^{K^*} = \frac{d_{L,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_K + \frac{d_{K,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_L, \tag{18}$$

which is the value obtained by the scheme described in [3] using the harmonic averaging points.
In (18), $d_{K,\sigma},\ d_{L,\sigma}$ are greater than 0 and denote the measures of the segments $[x_K, x_{K,\sigma}]$, $[x_L, x_{L,\sigma}]$, respectively. Two points $x_{K,\sigma},\ x_{L,\sigma}$ are the two orthogonal projection points of x_K and x_L onto σ.

Proof of property 4.1.2:

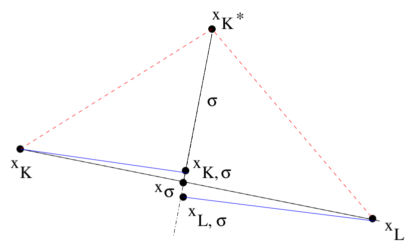


We present in detail the calculation of the coefficients of equation (17).

a) Calculation of the coefficient u_{K^*}:

$$
\begin{aligned}
\alpha_{K^*}^K + \alpha_{K^*}^L &= -\frac{n_{[x_\sigma,x_K]}.\ n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}} - \frac{n_{[x_\sigma,x_L]}.n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \\
&= \frac{m_{[x_\sigma,x_K]}.\ m_\sigma \cos\left(\varphi_2^{K,K^*}\right)}{m_{[x_\sigma,x_K]}.\ m_\sigma \sin\left(\varphi_2^{K,K^*}\right)} + \frac{m_{[x_\sigma,x_L]}.\ m_\sigma \cos\left(\varphi_2^{L,K^*}\right)}{m_{[x_\sigma,x_L]}.\ m_\sigma \sin\left(\varphi_2^{L,K^*}\right)} \\
&= \frac{\cos\left(\varphi_2^{K,K^*}\right)}{\sin\left(\varphi_2^{K,K^*}\right)} + \frac{\cos\left(\varphi_2^{L,K^*}\right)}{\sin\left(\varphi_2^{L,K^*}\right)} \\
&= \frac{\cos\left(\Pi-\varphi_2^{L,K^*}\right)}{\sin\left(\Pi-\varphi_2^{L,K^*}\right)} + \frac{\cos\left(\varphi_2^{L,K^*}\right)}{\sin\left(\varphi_2^{L,K^*}\right)} = 0.
\end{aligned}
$$

In the following figure, $x_{K,\sigma}$, $x_{L,\sigma}$ are the two orthogonal projection points of x_K, x_L on σ.



b) Calculation of the coefficient u_σ:

$$
\begin{aligned}
\alpha_\sigma^K + \alpha_\sigma^L &= \frac{n_{[x_{K^*},x_K]}.\ n^K_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_K,x_\sigma)}} + \frac{n_{[x_{K^*},x_L]}.n^L_{[x_\sigma,x_{K^*}]}}{2m_{(x_{K^*},x_L,x_\sigma)}} \\
&= \frac{m_{[x_{K^*},x_K]}.\ m_\sigma.\ \cos\left(\varphi_1^{K,K^*}\right)}{m_\sigma.\ d_{K,\sigma}} + \frac{m_{[x_{K^*},x_L]}.m_\sigma.\ \cos\left(\varphi_1^{L,K^*}\right)}{m_\sigma.\ d_{L,\sigma}} \\
&= -\frac{m_{[x_{K^*},x_{K,\sigma}]}}{d_{K,\sigma}} - \frac{m_{[x_{K^*},x_{L,\sigma}]}}{d_{L,\sigma}} \\
&= \frac{m_\sigma.(d_{K,\sigma}+d_{L,\sigma}) - m_{[x_{K,\sigma},x_\sigma]}.d_{K,\sigma} + m_{[x_{L,\sigma},x_\sigma]}.d_{L,\sigma}}{d_{K,\sigma}.d_{L,\sigma}} \\
&= -\frac{m_\sigma\,(d_{K,\sigma}+d_{L,\sigma})}{d_{K,\sigma}\,d_{L,\sigma}},
\end{aligned}
$$

because $m_{[x_{K^*},x_{K,\sigma}]} = m_{[x_{K^*},x_\sigma]} - m_{[x_{K,\sigma},x_\sigma]}, m_{[x_{K^*},x_{L,\sigma}]} = m_{[x_{K^*},x_\sigma]} + m_{[x_{L,\sigma},x_\sigma]}$, $m_{[x_{K,\sigma},x_\sigma]}.\ d_{L,\sigma} = m_{[x_{L,\sigma},x_\sigma]}.\ d_{K,\sigma}$.

c) Calculation of the coefficient u_K:

$$
\alpha_K = -\frac{\left(n^K_{[x_\sigma,x_{K^*}]}\right)^2}{2m_{(x_{K^*},x_K,x_\sigma)}} = -\frac{(m_\sigma)^2}{m_\sigma.\ d_{K,\sigma}} = -\frac{m_\sigma}{d_{K,\sigma}}.
$$

d) Calculation of the coefficient u_L:

$$\alpha_L = -\frac{\left(n^L_{[x_\sigma, x_{K^*}]}\right)^2}{2m_{(x_{K^*}, x_L, x_\sigma)}} = -\frac{(m_\sigma)^2}{m_\sigma d_{L,\sigma}} = -\frac{m_\sigma}{d_{L,\sigma}}.$$

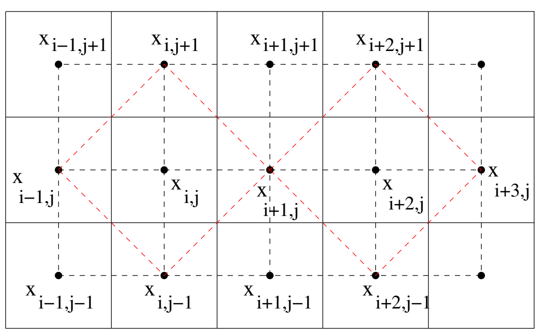
From the above calculations, we get:

$$\begin{aligned}\left(-\frac{m_\sigma\,(d_{K,\sigma}+d_{L,\sigma})}{d_{K,\sigma}\,d_{L,\sigma}}\right) u_\sigma^{K^*} &= -\frac{m_\sigma}{d_{K,\sigma}} u_K - \frac{m_\sigma}{d_{L,\sigma}} u_L.\\ u_\sigma^{K^*} &= \frac{d_{L,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_K + \frac{d_{K,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_L. \qquad \square\end{aligned}$$

Remark 4.2: In heterogeneous strongly anisotropic cases, the harmonic averaging point in [3] does not provide an "acceptable" interpolation u_σ. Even in this case, in Test 5 (see numerical results), we show that the FECC scheme can obtain precise results.

Remark 4.3: In property 4.1.2, we show a relationship between the FECC scheme and the scheme introduced in [3]. However, even in isotropic homogeneous cases, there is a difference between these two schemes.

For a grid of squares (the length of each edge is equal to a), we consider two control volumes $K_{i,j}$ (mesh point $x_{i,j}$) and $K_{i+1,j}$ (mesh point $x_{i+1,j}$) of the primary grid such that their edges do not belong to the boundary $\partial\Omega$. The third grid is defined in the following figure (dashed red and black lines):



Using the scheme described in [3], for a given constant function f ($\neq 0$), we get the two following linear equations at $x_{i,j}$ and $x_{i+1,j}$:

$$\begin{aligned}4u_{i,j} - u_{i,j+1} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} &= \int_{K_{i,j}} f dx = f.a^2,\\ 4u_{i+1,j} - u_{i,j} - u_{i+1,j+1} - u_{i+2,j} - u_{i+1,j-1} &= \int_{K_{i+1,j}} f dx = f.a^2.\end{aligned}$$

Using the FECC scheme, we get the two following linear equations at $x_{i,j}$ and $x_{i+1,j}$:

$$\begin{aligned} 4u_{i,j} - u_{i,j+1} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} &= \int\limits_{(x_{i,j+1};x_{i+1,j};x_{i,j-1};x_{i-1,j})} f.p_{i,j}dx = \frac{2.f.a^2}{3}, \\ 4u_{i+1,j} - u_{i,j} - u_{i+1,j+1} - u_{i+2,j} - u_{i+1,j-1} &= \int\limits_{(x_{i,j+1};x_{i+2,j+1};x_{i+2,j-1};x_{i,j-1})} f.p_{i+1,j}dx = \frac{4.f.a^2}{3}. \end{aligned}$$

At each point $x_{i,j}$ and $x_{i+1,j}$, we observe that the right hand sides of the two linear equations are different and that the left hand sides are the same.

5 Mathematical properties

We are interested in this section by the theoretical convergence of the FECC scheme in the general case where the tensor $\Lambda(x)$ can be discontinuous. We denote by $P_1(v)$ the traditional P_1 function on Ω, constructed on M^{**}. We first show the convergence of a variant of the original scheme, which we call FECCB, which satisfies the following discrete variational formulation:

$$u \in \mathcal{H}_{\mathcal{D}} \int\limits_{\Omega} \left(\Lambda(x)\nabla_{\mathcal{D},\Lambda}u(x)\right).\nabla_{\mathcal{D},\Lambda}v(x)\,dx = \int\limits_{\Omega} f(x)P_1(v)(x)dx \quad \text{for all } v \in \mathcal{H}_{\mathcal{D}}. \tag{19}$$

To simplify the presentation, we assume that, for neighboring control volumes, the line joining their primary mesh points intersects their common edge. Let $\mathcal{M}^{**}_{\Lambda} = \{K \in \mathcal{M}^{**} \mid \Lambda(x) \text{ is not continuous on } K\}$ and $\mathcal{M}^{**}_{Const} = \{K \in \mathcal{M}^{**} \mid \Lambda(x) \text{ is constant on } K\}$.
If $\forall K \in \mathcal{M}^{**}\backslash\mathcal{M}^{**}_{Const}$, let us assume that $\Lambda(x) = \Lambda_1$ on K_1 (K_1 is the triangle (x_K, x_σ, x_{K^*}) and that $\Lambda(x) = \Lambda_2$ on K_2 (K_2 is the triangle (x_L, x_σ, x_{K^*})) (see Figure 5.1). For $K \in \mathcal{M}^{**}_{\Lambda}$, we only choose two discontinuities to simplify the presentation but the method can be generalized to a greater number of discontinuities.
We denote by h_K the diameter of the triangle K and $\rho_K = sup\{diam(S) : \text{S is a ball contained in K}\}$.
As described in section 2, we recall that $\mathcal{V}^{**}_K$ is the set of the vertices of the triangle $K \in \mathcal{M}^{**}$. Moreover, the size of the discretization is defined by
$h_{\mathcal{M}^{**}} = sup\{h_K, K \in \mathcal{M}^{**}\}$.
For all the triangular cells belonging to $\mathcal{M}^{**}$, we join the centers of gravity of the triangles $x_{bar,K}$ to the midpoints of the edges $(x_{K,K^*}, x_{K,L}, x_{K^*,L})$. The vectors $\vec{\tau}_{K,K^*}$, $\vec{\tau}_{K^*,L}$, $\vec{\tau}_{L,K}$ are orthogonal vectors (with the same length) to the sides $[x_{K,K^*}, x_{bar,K}]$, $[x_{K^*,L}, x_{bar,K}]$, $[x_{K,L}, x_{bar,K}]$.
We denote by Ai_K, Ai_L and Ai_{K^*}, the polygons $(x_K, x_{K,L}, x_{bar,K}, x_{K,K^*})$, $(x_L, x_{K,L}, x_{bar,K}, x_{K^*,L})$ and $(x_{K^*}, x_{K^*,L}, x_{bar,K}, x_{K,K^*})$. The vectors $\vec{n}_K$, $\vec{n}_{K^*}$, $\vec{n}_L$, $\vec{n}_\sigma$, $\vec{n}_{K^*,1}$ and $\vec{n}_{K^*,2}$ are orthogonal vectors (with the same length) to the sides $[x_{K^*}, x_L]$, $[x_K, x_L]$, $[x_{K^*}, x_K]$, $[x_{K^*}, x_\sigma]$, $[x_K, x_\sigma]$ and $[x_\sigma, x_L]$ (see figure 5.1).
We define $\Pi^0_{\mathcal{D}^{**}}u$ which is a piecewise constant reconstruction by:
$\Pi^0_{\mathcal{D}^{**}}u(x) = \Pi^0_K u = u_K$ if $x \in Ai_K$, $\Pi^0_{\mathcal{D}^{**}}u(x) = \Pi^0_L u = u_L$ if $x \in Ai_L$ and

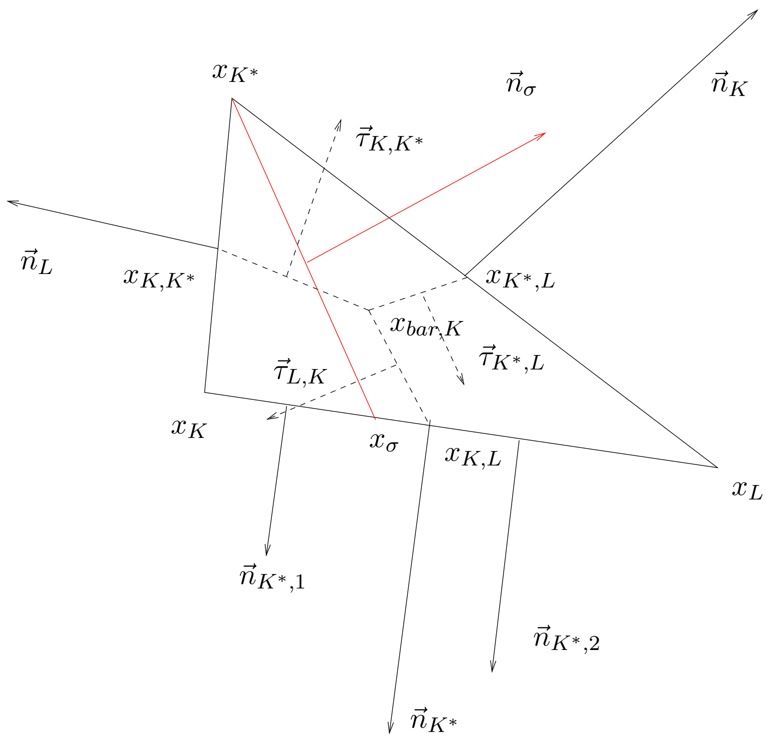


Figure 5.1

$\Pi^0_{\mathcal{D}^{**}}u(x) = \Pi^0_{K^*}u = u_{K^*}$ if $x \in Ai_{K^*}$.

We define the discrete H^1 norm of u by:

$$||u||^2_{1,\mathcal{D}^{**}} = \sum_{K\in\mathcal{M}^{**}} \frac{|\vec{\tau}_{K,L}|}{d(K,L)}(u_K-u_L)^2 + \frac{|\vec{\tau}_{K,K^*}|}{d(K,K^*)}(u_K-u_{K^*})^2 + \frac{|\vec{\tau}_{L,K^*}|}{d(L,K^*)}(u_L-u_{K^*})^2$$

where $d(K,L)$ is the distance between x_K and $[x_{K,L}, x_{bar,K}]$ ($d(K,L) = d(L,K)$), $d(K,K^*)$ is the distance between x_{K^*} and $[x_{K,K^*}, x_{bar,K}]$ ($d(K,K^*) = d(K^*,K)$) and $d(L,K^*)$ is the distance between x_L and $[x_{K^*,L}, x_{bar,K}]$ ($d(L,K^*) = d(K^*,L)$). Following the definition given in [18], we measure the strong consistency with the interpolation error function $S(\varphi) = \{||P_1(\varphi) - \varphi||^2_{L^2(\Omega)} + ||\nabla_{\mathcal{D},\Lambda}\varphi - \nabla\varphi||^2_{L^2(\Omega)^2}\}^{\frac{1}{2}}$, $\varphi \in [C_c^\infty(\Omega)]$ and the dual consistency with the conformity error function

$$W_{\mathcal{D}^{**}}(\vec{\varphi}) = max_{u\in\mathcal{H}_\mathcal{D}}||\frac{1}{\nabla_{\mathcal{D},\Lambda}u(x)}||\int_\Omega(\nabla_{\mathcal{D},\Lambda}u(x).\vec{\varphi}+P_1(u)(x)div\vec{\varphi}(x))dx, \forall\vec{\varphi}\in[C_c^\infty(\Omega)]^2.$$

Lemma 5.1**:** *With hypothesis 3.1, let $\mathcal{S}$ be a sequence of discretizations $\mathcal{D}^{**} = (\mathcal{H}_\mathcal{D}, h_{\mathcal{M}^{**}}, P_1(u), \nabla_{\mathcal{D},\Lambda})$ previously defined. We assume that there exists θ such that for all $\mathcal{D}^{**} \in \mathcal{S}$, for $K \in \mathcal{M}^{**}\backslash\{\mathcal{M}^{**}_\Lambda \cup \mathcal{M}^{**}_{Const}\}$,*
$\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{|\vec{n}_L|}\dfrac{|K_1|}{|K|} > \theta$ *and* $\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{|\vec{n}_K|}\dfrac{|K_2|}{|K|} > \theta.$
Then, for $K \in \mathcal{M}^{**}\backslash\mathcal{M}^{**}_\Lambda$ the gradient $\nabla_{\mathcal{D},\Lambda}u$ satisfies:
$|K|\nabla_{\mathcal{D},\Lambda}u =$

$(u_{K^*}-u_K)(\vec{\tau}_{K,K^*}+\vec{\epsilon}_{K,K^*})+(u_L-u_{K^*})(\vec{\tau}_{K^*,L}+\vec{\epsilon}_{K^*,L})+(u_K-u_L)(\vec{\tau}_{L,K}+\vec{\epsilon}_{L,K})$
with $lim_{h_{\mathcal{M}^{**}}\to 0}\dfrac{|\vec{\epsilon}_{K,K^*}|}{|\vec{\tau}_{K,K^*}|}=0$, $lim_{h_{\mathcal{M}^{**}}\to 0}\dfrac{|\vec{\epsilon}_{K^*,L}|}{|\vec{\tau}_{K^*,L}|}=0$ *and* $lim_{h_{\mathcal{M}^{**}}\to 0}\dfrac{|\vec{\epsilon}_{L,K}|}{|\vec{\tau}_{L,K}|}=0.$

Proof

First case: $\Lambda_1=\Lambda_2$.

Using the stokes formula, we obtain (see figure 5.1):

$$2|K|\nabla_{\mathcal{D},\Lambda}u=-u_{K^*}\vec{n}_{K^*}-u_L\vec{n}_L-u_K\vec{n}_K$$

which becomes

$$|K|\nabla_{\mathcal{D},\Lambda}=\frac{1}{6}\{(u_{K^*}-u_K)(\vec{n}_K-\vec{n}_{K^*})+(u_K-u_L)(\vec{n}_L-\vec{n}_K)+(u_L-u_{K^*})(\vec{n}_{K^*}-\vec{n}_L)\}.$$

As the vectors $\vec{\tau}_{L,K}$, $\vec{\tau}_{K,K^*}$ and $\vec{\tau}_{K^*,L}$ satisfy $\vec{\tau}_{L,K}=\dfrac{1}{6}(\vec{n}_L-\vec{n}_K)$,
$\vec{\tau}_{K,K^*}=\dfrac{1}{6}(\vec{n}_K-\vec{n}_{K^*})$, $\vec{\tau}_{K^*,L}=\dfrac{1}{6}(-\vec{n}_L+\vec{n}_{K^*})$, we obtain:

$$|K|\nabla_{\mathcal{D},\Lambda}u=(u_{K^*}-u_K)\vec{\tau}_{K,K^*}+(u_L-u_{K^*})\vec{\tau}_{K^*,L}+(u_K-u_L)\vec{\tau}_{L,K}.$$

Second case: $\Lambda_1\neq\Lambda_2$.

For $x\in K_1$, the gradient $\nabla_{\mathcal{D},\Lambda}u$ satisfies:

$$2|K_1|\nabla_{\mathcal{D},\Lambda}u=-u_{K^*}\vec{n}_{K^*,1}-u_\sigma\vec{n}_L-u_K\vec{n}_\sigma.$$

We can write $u_\sigma=\beta_K u_K+\beta_L u_L+\beta_{K^*}u_{K^*}$ with $\beta_K+\beta_L+\beta_{K^*}=1$,
$lim_{h_{\mathcal{M}^{**}}\to 0}\beta_{K^*}=0$, $lim_{h_{\mathcal{M}^{**}}\to 0}\beta_K=\dfrac{[x_L,x_\sigma]}{[x_L,x_K]}$ and $lim_{h_{\mathcal{M}^{**}}\to 0}\beta_L=\dfrac{[x_K,x_\sigma]}{[x_L,x_K]}$ because, for $K\in\mathcal{M}^{**}\backslash\{\mathcal{M}^{**}_\Lambda\cup\mathcal{M}^{**}_{Const}\}$ ($\Lambda(x)$ is continuous), $lim_{h_{\mathcal{M}^{**}}\to 0}||\Lambda_1-\Lambda_2||=0$.

The gradient becomes:

$$2|K_1|\nabla_{\mathcal{D},\Lambda}u=-u_{K^*}(\vec{n}_{K^*,1}+\epsilon_{K^*}\vec{n}_L)-u_L(\frac{[x_K,x_\sigma]}{[x_L,x_K]}+\epsilon_L)\vec{n}_L-u_K(\vec{n}_\sigma+\vec{n}_L(\epsilon_K+\frac{[x_L,x_\sigma]}{[x_L,x_K]})).$$

with $\epsilon_K+\epsilon_L+\epsilon_{K^*}=0$, $lim_{h_{\mathcal{M}^{**}}\to 0}\epsilon_{K^*}=0$, $lim_{h_{\mathcal{M}^{**}}\to 0}\epsilon_K=0$ and $lim_{h_{\mathcal{M}^{**}}\to 0}\epsilon_L=0$.
As $\dfrac{(\vec{n}_\sigma+\frac{[x_L,x_\sigma]}{[x_L,x_K]}\vec{n}_L)}{|K_1|}=\dfrac{\vec{n}_K}{|K|}$, $\dfrac{\frac{[x_K,x_\sigma]}{[x_L,x_K]}\vec{n}_L}{|K_1|}=\dfrac{\vec{n}_L}{|K|}$ and $\dfrac{\vec{n}_{K^*,1}}{|K_1|}=\dfrac{\vec{n}_{K^*}}{|K|}$, we deduce:
$2|K|\nabla_{\mathcal{D},\Lambda}u=-u_{K^*}(\vec{n}_{K^*}+\dfrac{|K|}{|K_1|}\epsilon_{K^*}\vec{n}_L)-u_L(1+\dfrac{|K|}{|K_1|}\epsilon_L)\vec{n}_L-u_K(\vec{n}_K+\dfrac{|K|}{|K_1|}\epsilon_K\vec{n}_L)$
$=-u_{K^*}\vec{p}_{K^*}-u_L\vec{p}_L-u_K\vec{p}_K.$
Finally, we obtain:

$$|K|\nabla_{\mathcal{D},\Lambda}u=\frac{1}{6}\{(u_{K^*}-u_K)(\vec{p}_K-\vec{p}_{K^*})+(u_K-u_L)(\vec{p}_L-\vec{p}_K)+(u_L-u_{K^*})(\vec{p}_{K^*}-\vec{p}_L)\}.$$

Using the assumption on the grid ($\dfrac{min(|\vec{\tau}_{K,L}|,|\vec{\tau}_{K,K^*}|,|\vec{\tau}_{K,K^*}|)}{|\vec{n}_L|}\dfrac{|K_1|}{|K|}>\theta$), choosing
$\vec{\epsilon}_{K,K^*}=-\frac{|K|}{|K_1|}\epsilon_{K^*}\vec{n}_L+\frac{|K|}{|K_1|}\epsilon_K\vec{n}_L$, $\vec{\epsilon}_{K^*,L}=\frac{|K|}{|K_1|}\epsilon_{K^*}\vec{n}_L-\frac{|K|}{|K_1|}\epsilon_L\vec{n}_L$,

$\vec{\epsilon}_{L,K} = \frac{|K|}{|K_1|}\epsilon_L \vec{n}_L - \frac{|K|}{|K_1|}\epsilon_K \vec{n}_L$ and using the values of $\vec{\tau}_{L,K}$,$\vec{\tau}_{K,K^*}$ and $\vec{\tau}_{K^*,L}$, we obtain the desired property for $x \in K_1$.
For $x \in K_2$, the computation of the gradient is similar.

□

Lemma 5.2**:** *With hypothesis 3.1, let $\mathcal{S}$ be a sequence of discretization $\mathcal{D}^{**} = (\mathcal{H}_{\mathcal{D}}, h_{\mathcal{M}^{**}}, P_1(u), \nabla_{\mathcal{D},\Lambda})$ previously defined. We assume that there exists θ, such that for all $\mathcal{D}^{**} \in \mathcal{S}$, for $K \in \mathcal{M}^{**}_{\Lambda}$*

- $|\dfrac{\vec{n}_\sigma \Lambda_2 \vec{n}_K}{|K_2|} - \dfrac{\vec{n}_\sigma \Lambda_1 \vec{n}_L}{|K_1|}| \geq \theta(\dfrac{\vec{n}_\sigma \Lambda_1 \vec{n}_\sigma}{|K_1|} + \dfrac{\vec{n}_\sigma \Lambda_2 \vec{n}_\sigma}{|K_2|})$, *($H_1$)*
- $\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{max(|\vec{n}_\sigma|, |\vec{n}_{K^*,1}|, |\vec{n}_{K^*,2}|)} > \theta$, *($H_2$).*

Then, there exists a constant C_2 such that the gradients $\nabla_{K_1} u$ and $\nabla_{K_2} u$ satisfy:

$$|K_1|\nabla_{K_1} u = (u_{K^*} - u_K)\vec{\theta}_1(K, K^*) + (u_L - u_{K^*})\vec{\theta}_1(K^*, L) + (u_K - u_L)\vec{\theta}_1(L, K)$$

and

$$|K_2|\nabla_{K_2} u = (u_{K^*} - u_K)\vec{\theta}_2(K, K^*) + (u_L - u_{K^*})\vec{\theta}_2(K^*, L) + (u_K - u_L)\vec{\theta}_2(L, K).$$

with $|\vec{\theta}_1(K, K^*)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{K,K^*}|$, $|\vec{\theta}_1(K^*, L)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{K^*,L}|$, $|\vec{\theta}_1(L, K)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{L,K}|$, $|\vec{\theta}_2(K, K^*)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{K,K^*}|$, $|\vec{\theta}_2(K^*, L)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{K^*,L}|$ *and* $|\vec{\theta}_2(L, K)| \leq \dfrac{C_2}{\theta^2}|\vec{\tau}_{L,K}|$.
Remark 5.1**:** In order to satisfy assumption H_1, it is sufficient to choose the primary mesh points x_K and x_L close enough from x_σ. In pratice, this is a light hypothesis.

Proof
Let us denote $Det = -\dfrac{\vec{n}_\sigma \Lambda_1 \vec{n}_L}{|K_1|} + \dfrac{\vec{n}_\sigma \Lambda_2 \vec{n}_K}{|K_2|}$.
We can write:

$$u_\sigma = \beta_K u_K + \beta_L u_L + \beta_{K^*} u_{K^*} \tag{20}$$

with $\beta_K + \beta_L + \beta_{K^*} = 1$ where $\beta_K = \dfrac{\vec{n}_\sigma \Lambda_1 \vec{n}_\sigma}{Det|K_1|}$, $\beta_L = \dfrac{\vec{n}_\sigma \Lambda_2 \vec{n}_\sigma}{Det|K_2|}$ and $\beta_{K^*} = 1 - \beta_K - \beta_L$.
Using assumption H_1, we obtain $|Det| \geq \theta(\dfrac{\vec{n}_\sigma \Lambda_1 \vec{n}_\sigma}{|K_1|} + \dfrac{\vec{n}_\sigma \Lambda_2 \vec{n}_\sigma}{|K_2|})$. We deduce $|\beta_K| \leq \dfrac{1}{\theta}$, $|\beta_L| \leq \dfrac{1}{\theta}$ and $\beta_{K^*} \leq \dfrac{2}{\theta} + 1$. Using the formula:

$$2|K_1|\nabla_{K_1} u = -(u_{K^*} - u_\sigma)\vec{n}_{K^*,1} - (u_K - u_\sigma)\vec{n}_\sigma,$$

$$2|K_2|\nabla_{K_2} u = -(u_{K^*} - u_\sigma)\vec{n}_{K^*,2} - (u_L - u_\sigma)\vec{n}_\sigma$$

and equality (20), we obtain: $\vec{\theta}_1(K, K^*) = -\dfrac{\beta_K}{2}\vec{n}_{K^*,1} + \dfrac{\beta_{K^*}}{2}\vec{n}_\sigma$,
$\vec{\theta}_1(K^*, L) = \dfrac{\beta_L}{2}\vec{n}_{K^*,1}$, $\vec{\theta}_1(L, K) = -\dfrac{\beta_L}{2}\vec{n}_\sigma$, $\vec{\theta}_2(K, K^*) = -\dfrac{\beta_K}{2}\vec{n}_{K^*,2}$,

$\vec{\theta}_2(K^*, L) = \dfrac{\beta_L}{2}\vec{n}_{K^*,2} - \dfrac{\beta_{K^*}}{2}\vec{n}_\sigma$ and $\vec{\theta}_2(L, K) = \dfrac{\beta_L}{2}\vec{n}_\sigma$.
We conclude using assumption H_2. □

Proposition 5.3**:** *With hypothesis 3.1, let $\mathcal{S}$ be a sequence of discretizations $\mathcal{D}^{**} = (\mathcal{H}_\mathcal{D}, h_{\mathcal{M}^{**}}, P_1(u), \nabla_{\mathcal{D},\Lambda})$ previously defined. We assume that there exists θ, such that for all $\mathcal{D}^{**} \in \mathcal{S}$:*

- $\rho_K > \theta h_K \ \forall K \in \mathcal{M}^{**}_{Const}$ *(H_1),*
- $\rho_{K_1} > \theta h_{K_1} \ \forall K \in \mathcal{M}^{**}\backslash\mathcal{M}^{**}_{Const}$ *(H_2),*
- $\rho_{K_2} > \theta h_{K_2} \ \forall K \in \mathcal{M}^{**}\backslash\mathcal{M}^{**}_{Const}$ *(H_3),*
- $d(K,L) > \dfrac{1}{\theta}|\vec{\tau}_{K,L}|,\ d(K,K^*) > \dfrac{1}{\theta}|\vec{\tau}_{K,K^*}|,\ d(L,K^*) > \dfrac{1}{\theta}|\vec{\tau}_{L,K^*}|,\ \forall K \in \mathcal{M}^{**}$ *(H_4),*
- $|\dfrac{\vec{n}_\sigma\Lambda_2\vec{n}_K}{|K_2|} - \dfrac{\vec{n}_\sigma\Lambda_1\vec{n}_L}{|K_1|}| \geq \theta(\dfrac{\vec{n}_\sigma\Lambda_1\vec{n}_\sigma}{|K_1|} + \dfrac{\vec{n}_\sigma\Lambda_2\vec{n}_\sigma}{|K_2|})$ *for* $K \in \mathcal{M}^{**}_\Lambda$ *(H_5),*
- $\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{max(|\vec{n}_\sigma|, |\vec{n}_{K^*,1}|, |\vec{n}_{K^*,2}|)} > \theta$ *for* $K \in \mathcal{M}^{**}_\Lambda$ *(H_6),*
- $\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{|\vec{n}_L|}\dfrac{|K_1|}{|K|} > \theta$ *and*
  $\dfrac{min(|\vec{\tau}_{K,L}|, |\vec{\tau}_{K,K^*}|, |\vec{\tau}_{K,K^*}|)}{|\vec{n}_K|}\dfrac{|K_2|}{|K|} > \theta$ *for* $K \in \mathcal{M}^{**}\backslash\{\mathcal{M}^{**}_\Lambda \cup \mathcal{M}^{**}_{Const}\}$ *(H_7).*

Then, the FECCB scheme is coercive, that is to say there exists $C_{\mathcal{D}^{**}}$ such that

$$||P_1(u)||_{L^2(\Omega)} \leq C_{\mathcal{D}^{**}}||\nabla_{\mathcal{D},\Lambda}u||,\ u \in \mathcal{H}_\mathcal{D}.$$

Moreover $\forall\varphi \in [C_c^\infty(\Omega)]$, $lim_{h_{\mathcal{M}^{**}}\to 0}S(\varphi) = 0$ and $\forall\vec{\varphi} \in [C_c^\infty(\Omega)]^2$ $lim_{h_{\mathcal{M}^{**}}\to 0}W_{\mathcal{D}^{**}}(\vec{\varphi}) = 0$. With these three properties, we can apply the corollary 2.3 described in [18]. It means that the FECCB scheme is convergent, that is to say, $P_1(u)$ converges to the exact solution u_{exa} of the problem and $\nabla_{\mathcal{D},\Lambda}u$ tends to ∇u_{exa} as $h_{\mathcal{M}^{**}} \to 0$.

Proof

Following Lemma 5.3 in [16], there exists C_3 only depending on θ (Poincaré inequality) such that:

$$||\Pi^0_{\mathcal{D}^{**}}u||^2_{L^2(\Omega)} \leq C_3||u||^2_{1,\mathcal{D}^{**}}, \forall u \in \mathcal{H}_\mathcal{D}.$$

Let us show that there exists C_4 only depending on Ω and θ such that:

$$||u||^2_{1,\mathcal{D}^{**}} \leq C_4||\nabla_{\mathcal{D},\Lambda}u(x)||^2_{\{L^2(\Omega)\}^2}. \tag{21}$$

The proof is close to the one described in [18] (lemma 3.1). We denote by $\nabla_K u$ the value of $\nabla_{\mathcal{D},\Lambda} u(x)$ if $K \in \mathcal{M}^{**}_{Const}$, and $\nabla_{K_1} u$ (resp. $\nabla_{K_2} u$) the value of $\nabla_{\mathcal{D},\Lambda} u(x)$ on K_1 (resp. K_2) if $K \in \mathcal{M}^{**} \backslash \mathcal{M}^{**}_{Const}$. For all $K \in \mathcal{M}^{**}_{Const}$, for all $s \in \mathcal{V}^{**}_K$ and $r \in \mathcal{V}^{**}_K$, we can write:

$$u_s - u_r = \nabla_K u.(s-r).$$

We obtain:

$$||\nabla_K u||^2 \geq \frac{1}{3}(\frac{(u_K-u_L)^2}{[x_K,x_L]^2} + \frac{(u_K-u_{K^*})^2}{[x_K,x_{K^*}]^2} + \frac{(u_L-u_{K^*})^2}{[x_L,x_{K^*}]^2})$$

and

$$|K|||\nabla_K u||^2 \geq \frac{1}{3}((u_K-u_L)^2 \frac{\rho_K}{[x_K,x_L]} + (u_K-u_{K^*})^2 \frac{\rho_K}{[x_K,x_{K^*}]} + (u_L-u_{K^*})^2 \frac{\rho_K}{[x_L,x_{K^*}]}). \tag{22}$$

For $K \in \mathcal{M}^{**} \backslash \mathcal{M}^{**}_{Const}$, we get:

$$||\nabla_{K_1} u||^2 \geq \frac{1}{2}(\frac{(u_K-u_\sigma)^2}{[x_K,x_\sigma]^2} + \frac{(u_K-u_{K^*})^2}{[x_K,x_{K^*}]^2})$$

and

$$||\nabla_{K_2} u||^2 \geq \frac{1}{2}(\frac{(u_L-u_\sigma)^2}{[x_\sigma,x_L]^2} + \frac{(u_L-u_{K^*})^2}{[x_L,x_{K^*}]^2}).$$

We deduce that there exists C_5 such that:

$$|K_1|||\nabla_{K_1} u||^2 + |K_2|||\nabla_{K_2} u||^2 \geq C_5\{(min(\frac{\rho_{K_1}}{[x_K,x_\sigma]}, \frac{\rho_{K_2}}{[x_L,x_\sigma]})(u_K-u_\sigma)^2 + (u_L-u_\sigma)^2) + \frac{\rho_{K_1}}{[x_K,x_{K^*}]}(u_K-u_{K^*})^2 + \frac{\rho_{K_2}}{[x_L,x_{K^*}]}(u_L-u_{K^*})^2\}. \tag{23}$$

Using the inequality $(u_K-u_\sigma)^2 + (u_L-u_\sigma)^2 \geq \dfrac{1}{2}(u_K-u_L)^2$, we obtain that there exists C_{21} such that:
$\sum_{K\in\mathcal{M}^{**}\backslash\mathcal{M}^{**}_{Const}} |K_1|||\nabla_{K_1} u||^2 + |K_2|||\nabla_{K_2} u||^2 \geq$
$C_{21} \sum_{K\in\mathcal{M}^{**}\backslash\mathcal{M}^{**}_{Const}} min(\dfrac{\rho_{K_1}}{[x_K,x_\sigma]}, \dfrac{\rho_{K_2}}{[x_L,x_\sigma]})(u_K-u_L)^2 + \dfrac{\rho_{K_1}}{[x_K,x_{K^*}]}(u_K-u_{K^*})^2 + \dfrac{\rho_{K_2}}{[x_L,x_{K^*}]}(u_L-u_{K^*})^2$.
Using assumptions H_1, H_2, H_3 and inequality (22), there exists C_{23} such that:

$$||\nabla_{\mathcal{D},\Lambda} u(x)||^2 \geq C_{23} \sum_{K\in\mathcal{M}^{**}} \theta(u_K-u_L)^2 + \theta(u_K-u_{K^*})^2 + \theta(u_L-u_{K^*})^2.$$

Using assumption H_4, we get:

$$||\nabla_{\mathcal{D},\Lambda} u(x)||^2 \geq C_{23} \sum_{K\in\mathcal{M}^{**}} \frac{|\vec{\tau}_{K,L}|}{d(K,L)}(u_K-u_L)^2 + \frac{|\vec{\tau}_{K,K^*}|}{d(K,K^*)}(u_K-u_{K^*})^2 + \frac{|\vec{\tau}_{L,K^*}|}{d(L,K^*)}(u_L-u_{K^*})^2,$$

which is the desired inequality.
Moreover, for $K \in \mathcal{M}^{**}$, we write:
if $x \in Ai_K$, $P_1(u)(x) = \Pi^0_{\mathcal{D}^{**}} u(x) + \nabla_{P_{1,K}} u.(x - x_K)$, if $x \in Ai_L$,
$P_1(u)(x) = \Pi^0_{\mathcal{D}^{**}} u(x) + \nabla_{P_{1,L}} u.(x - x_L)$, if $x \in Ai_{K^*}$, $P_1(u)(x) = \Pi^0_{\mathcal{D}^{**}} u(x) + \nabla_{P_{1,K^*}} u.(x - x_{K^*})$. We obtain that

$$||P_1(u)||_{L^2(\Omega)} \leq ||\Pi^0_{\mathcal{D}^{**}} u||_{L^2(\Omega)} + h_{\mathcal{M}^{**}} ||\nabla_{P_1} u(x)||_{\{L^2(\Omega)\}^2}. \tag{24}$$

With Lemma 5.1, we obtain the formulation:

$$|K|\nabla_{P_1} u = (u_K - u_{K^*})\vec{\tau}_{K,K^*} + (u_{K^*} - u_L)\vec{\tau}_{K^*,L} + (u_L - u_K)\vec{\tau}_{L,K} \; for \; x \in K, \; for \; K \in \mathcal{M}^{**}.$$

We define $S_K = ((K,L),(K,K^*),(K^*,L))$. Using the Cauchy-Schwarz inequality and $\sum_{K \in \mathcal{M}^{**}} \sum_{(M,N) \in S_K} |\vec{\tau}_{M,N}| d(M,N) = 2|\Omega|$, we get that there exists C_{20} such that:

$$||\nabla_{P_1} u||_{\{L^2(\Omega)\}^2} \leq C_{20} ||u||_{1,\mathcal{D}^{**}}. \tag{25}$$

Using (21), (24) and (25), we obtain that there exists C_{22} such that:

$$||P_1(u)||_{L^2(\Omega)} \leq C_{22} ||\nabla_{\mathcal{D},\Lambda} u(x)||_{\{L^2(\Omega)\}^2}. \tag{26}$$

We conclude that the scheme is coercive.

Let us estimate the strong consistency of the discretization. Let $\varphi \in [C_c^\infty(\Omega)]$. As we use the Stokes formula to approximate the gradient, with assumptions H_1, H_2 and H_3, we obtain in the same way as lemma 4.3 described in [16] that there exists C_6 only depending on θ and φ such that

$$||\nabla_{\mathcal{D},\Lambda} \varphi - \nabla \varphi||_{L^2(\Omega)^2} \leq C_6 h_{M^{**}}.$$

Moreover, using Lemma 3.1 in [19], we obtain that $||P_1(\varphi) - \varphi||_{L^2(\Omega)} \leq h_{M^{**}} ||\nabla \varphi||_{L^2(\Omega)^2}$. We deduce that the interpolation error function $S(\varphi)$ tends toward zero if $h_{M^{**}}$ tends toward zero.

Let $\vec{\varphi} \in [C_c^\infty(\Omega)]^2$. Let us compute $T = \int_\Omega (\nabla_{\mathcal{D},\Lambda} u(x).\vec{\varphi} + P_1(u) div \vec{\varphi}(x)) dx$.
We denote by $\vec{\varphi}_K$ the average value of $\vec{\varphi}(x)$ if $K \in \mathcal{M}^{**} \backslash \mathcal{M}^{**}_\Lambda$, $\vec{\varphi}_{K_1}$ (resp. $\vec{\varphi}_{K_2}$) the average value of $\vec{\varphi}(x)$ on K_1 (resp. K_2) if $K^{**} \in \mathcal{M}^{**}_\Lambda$, $\vec{\varphi}_{M,N(M,N) \in S_K}$ the average value of $\vec{\varphi}(x)$ on $\vec{\tau}_{M,N,(M,N) \in S_K}$.
We get $T = T_1 + T_2 + T_3$ with
$T_1 = \sum_{K \in \mathcal{M}^{**}_{Const}} |K| \nabla_K u.\vec{\varphi}_K + \sum_{K \in \mathcal{M}^{**} \backslash \{\mathcal{M}^{**}_{Const} \cup \mathcal{M}^{**}_\Lambda\}} |K_1| \nabla_{K_1} u.\vec{\varphi}_{K_1} + |K_2| \nabla_{K_2} u.\vec{\varphi}_{K_2^{**}} +$

$\sum_{K \in \mathcal{M}^{**}_\Lambda} |K_1| \nabla_{K_1} u.\vec{\varphi}_{K_1} + |K_2| \nabla_{K_2} u.\vec{\varphi}_{K_2^{**}} = T_{11} + T_{12} + T_{13} + T_{14} + T_{15}$,

$$T_2 = \sum_{K \in \mathcal{M}^{**}} (u_K - u_{K^*})\vec{\tau}_{K,K^*}.\vec{\varphi}_{K,K^*} + (u_{K^*} - u_L)\vec{\tau}_{K^*,L}.\vec{\varphi}_{K^*,L} + (u_L - u_K)\vec{\tau}_{L,K}.\vec{\varphi}_{K,L},$$

$T_3 = \sum_{K \in \mathcal{M}^{**}} \int_{Ai_K} \nabla_{P_{1,K}} u.(x - x_K) div \vec{\varphi}(x) dx + \int_{Ai_L} \nabla_{P_{1,L}} u.(x - x_L) div \vec{\varphi}(x) dx +$

$\int_{Ai_{K^*}} \nabla_{P_{1,K^*}} u.(x - x_{K^*}) div\vec{\varphi}(x)dx.$
We obtain:

$$|T_3| \leq h^{**}_{\mathcal{M}} ||\nabla_{P_1} u||_{\{L^2(\Omega)\}^2} ||div\vec{\varphi}||_{L^2(\Omega)}. \qquad (27)$$

With assumption H_7 and Lemma 5.1, we get:

$T_{11} + T_{12} + T_{13} = \sum_{K \in \mathcal{M}^{**} \backslash \mathcal{M}^{**}_{\Lambda}} \{(u_{K^*} - u_K)(\vec{\tau}_{K,K^*} + \vec{\epsilon}_{K,K^*}) +$
$(u_L - u_{K^*})(\vec{\tau}_{K^*,L} + \vec{\epsilon}_{K^*,L}) + (u_K - u_L)(\vec{\tau}_{L,K} + \vec{\epsilon}_{L,K})\}.\vec{\varphi}_K.$

On the other hand, with Lemma 5.2, we write:

$$T_{14} = \sum_{K \in \mathcal{M}^{**}_{\Lambda}} ((u_{K^*} - u_K)\vec{\theta}_1(K, K^*) + (u_L - u_{K^*})\vec{\theta}_1(K^*, L) + (u_K - u_L)\vec{\theta}_1(L, K)).\vec{\varphi}_{K_1}$$

and

$$T_{15} = \sum_{K \in \mathcal{M}^{**}_{\Lambda}} ((u_{K^*} - u_K)\vec{\theta}_2(K, K^*) + (u_L - u_{K^*})\vec{\theta}_2(K^*, L) + (u_K - u_L)\vec{\theta}_2(L, K)).\vec{\varphi}_{K_2}.$$

Let us define

$$Term_1 = \sum_{K \in \mathcal{M}^{**} \backslash \mathcal{M}^{**}_{\Lambda}} \sum_{(M,N) \in S_K} (|\vec{\tau}_{M,N}| + |\vec{\epsilon}_{M,N})|)d(M, N)(|\vec{\varphi}_K - \vec{\varphi}_{M,N}|^2)$$

and

$$Term_2 = \sum_{K \in \mathcal{M}^{**}_{\Lambda}} \sum_{(M,N) \in S_K} |\vec{\tau}_{M,N}| d(M, N)(|\frac{\vec{\theta}_1(M, N).\vec{\varphi}_{K_1}}{|\vec{\tau}_{M,N}|} + \frac{\theta_2(M, N)\vec{\varphi}_{K_2}}{|\vec{\tau}_{M,N}|} - \vec{\varphi}_{M,N}|^2).$$

Using the Cauchy-Schwarz inequality, we obtain:
$|T_1 + T_2|^2 \leq ||u||^2_{1,\mathcal{D}^{**}} (Term_1 + Term_2).$
Besides, using the regularity of $\vec{\varphi}$, there exists a constant C_ϕ such that $\forall (M, N) \in S_K$ $|\varphi_K - \varphi_{M,N}|^2 \leq C_\phi h^2_{\mathcal{M}^{**}}$. Using $\sum_{K \in \mathcal{M}^{**}} \sum_{(M,N) \in S_K} |\vec{\tau}_{M,N}| d(M, N) = 2|\Omega|$, and Lemma 5.1, there exists C_{30} such that
$Term_1 \leq C_{30} |\Omega| (h^2_{\mathcal{M}^{**}} (1 + \epsilon_1(h_{\mathcal{M}^{**}})))$ with $lim_{h_{\mathcal{M}^{**}} \to 0} \epsilon_1(h_{\mathcal{M}^{**}}) = 0$.
Moreover, using the regularity of $\vec{\varphi}$, assumptions H_5, H_6 and Lemma 5.2, we get that there exists C_{10} depending on θ and $\vec{\varphi}$ such that

$$\forall (M, N) \in S_K \ (|\frac{\vec{\theta}_1(M, N).\vec{\varphi}_{K_1}}{|\vec{\tau}_{M,N}|} + \frac{\theta_2(M, N)\vec{\varphi}_{K_2}}{|\vec{\tau}_{M,N}|} - \vec{\varphi}_{M,N}|^2) \leq C_{10}.$$

Following the same arguments as those described in [12] (Theorem 3.8), as the tensor $\lambda(x)$ is piecewise Lipschitz-continuous, we deduce that $\sum_{K \in \mathcal{M}^{**}_{\Lambda}} \sum_{(M,N) \in S_K} |\vec{\tau}_{M,N}| d(M, N)$ tends toward zero if $h_{\mathcal{M}^{**}}$ tends toward zero. (It means that the dimension of the zones where the tensor $\lambda(x)$ is discontinuous is inferior to one.)
We finally obtain that there exists C_8 such that:

$$|T_1 + T_2| \leq C_8 ||u||_{1,\mathcal{D}^{**}} (h_{\mathcal{M}^{**}} + \epsilon_2(h_{\mathcal{M}^{**}}))$$

with $lim_{h_{\mathcal{M}^{**}}\to 0}\epsilon_2(h_{\mathcal{M}^{**}}) = 0$. Using (21), (27) and (25), we deduce that there exists C_9 only depending on θ and $\vec{\varphi}$ such that $W_{\mathcal{D}^{**}}(\vec{\varphi}) \le C_9(h_{\mathcal{M}^{**}} + \epsilon(h_{\mathcal{M}^{**}}))$. This is the dual consistency described in [18]. □

Corollary 5.4**:** *With hypothesis 3.1, let $\mathcal{S}$ be a sequence of discretizations $\mathcal{D}^{**} = (\mathcal{H}_{\mathcal{D}}, h_{\mathcal{M}^{**}}, P(u), \nabla_{\mathcal{D},\Lambda})$ defined in (3). Under the assumptions of Proposition 5.3, the FECC scheme is convergent, that is to say, $P(u)$ converges to the exact solution u_{exa} of the problem and $\nabla_{\mathcal{D},\Lambda}u$ tends to ∇u_{exa} as $h_{\mathcal{M}^{**}} \to 0$.*

Proof

We measure the strong consistency of the FECC scheme with the interpolation error function $S_i(\varphi) = \{||P(\varphi) - \varphi||^2_{L^2(\Omega)} + ||\nabla_{\mathcal{D},\Lambda}\varphi - \nabla\varphi||^2_{L^2(\Omega)^2}\}^{\frac{1}{2}}$, $\varphi \in [C_c^\infty(\Omega)]$ and the dual consistency with the conformity error function

$$W_{i,\mathcal{D}^{**}}(\vec{\varphi}) = max_{u\in\mathcal{H}_{\mathcal{D}}}||\frac{1}{\nabla_{\mathcal{D},\Lambda}u(x)}||\int_\Omega(\nabla_{\mathcal{D},\Lambda}u(x).\vec{\varphi}+P(u)(x)div\vec{\varphi}(x))dx, \forall\vec{\varphi} \in [C_c^\infty(\Omega)]^2.$$

Using the definition of $P(u)$ and $P_1(u)$, we obtain that:

$$||P(u) - P_1(u)||_{L^2(\Omega)} \le h_{\mathcal{M}^{**}}(||\nabla_{P_1}u||_{L^2(\Omega)^2} + ||\nabla_{\mathcal{D},\Lambda}u||_{L^2(\Omega)^2}).$$

With (21) and (25), there exists C_{40} such that:

$$||P(u) - P_1(u)||_{L^2(\Omega)} \le C_{40}h_{\mathcal{M}^{**}}||\nabla_{\mathcal{D},\Lambda}u||_{L^2(\Omega)^2}. \tag{28}$$

In the same way, for $\varphi \in [C_c^\infty(\Omega)]$, we get that:

$$||P(\varphi) - P_1(\varphi)||_{L^2(\Omega)} = \epsilon_3(h_{\mathcal{M}^{**}}) \tag{29}$$

with $lim_{h_{\mathcal{M}^{**}}\to 0}\epsilon_3(h_{\mathcal{M}^{**}}) = 0$. Using (26), we deduce that there exists C_{41} such that

$$||P(u)||_{L^2(\Omega)} \le C_{41}||\nabla_{\mathcal{D},\Lambda}u(x)||_{\{L^2(\Omega)\}^2}. \tag{30}$$

We deduce that the FECC scheme is coercive. Using proposition 5.3, (28) and (29), we obtain that $\forall\varphi \in [C_c^\infty(\Omega)]$, $lim_{h_{\mathcal{M}^{**}}\to 0}S_i(\varphi) = 0$ and $\forall\vec{\varphi} \in [C_c^\infty(\Omega)]^2$ $lim_{h_{\mathcal{M}^{**}}\to 0}W_{i,\mathcal{D}^{**}}(\vec{\varphi}) = 0$. We conclude applying corollary 2.3 described in [18]. □

6 Numerical results

We introduce some notations for all the tests:

- nunkw: number of unknowns,
- umin: value of the minimum of the approximate solution,
- umax: value of the maximum of the approximate solution.

Let us denote by u_{ana} the exact solution, $u_{\mathcal{M}} = (u_K)_{K\in\mathcal{M}}$ the piecewise constant approximate solution,

- erl2, the relative discrete L^2 norm of the error, as follows:

$$\text{erl2} = \left(\frac{\sum\limits_{K\in\mathcal{M}} |K|(u_{ana}(x_K) - u_K)^2}{\sum\limits_{K\in\mathcal{M}} |K| u_{ana}(x_K)^2} \right)^{\frac{1}{2}} ,$$

- ergrad, the relative L^2 norm of the error on the gradient,
- ratiol2: for $i \geq 2$,

$$\text{ratiol2(i)} = -2 \frac{\ln(\text{erl2(i)}) - \ln(\text{erl2(i - 1)})}{\ln(\text{nunkw(i)}) - \ln(\text{nunkw(i - 1)})} ,$$

- ratiograd, for $i \geq 2$, the same formula as above with ergrad instead of erl2.

Test 1: Mild anisotropy
We consider an homogeneous anisotropic tensor, as follows:

$$\Lambda = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} .$$

Test 1.1 The exact solution u_{ana} and the source term f satisfy:

$$\begin{cases} u_{\text{ana}}(x, y) = 16x(1 - x)y(1 - y) \quad \text{in} \quad (0, 1) \times (0, 1) \,, \\ u_{\text{ana}}(x, \text{y}) = 0 \quad \text{on the boundary of } [0, 1] \times [0, 1], \\ f(x, y) = -\nabla. \left(\Lambda \nabla u_{\text{ana}}\right) . \end{cases}$$

nunkw	erl2	ratiol2	umin	umax	ergrad	ratiograd
56	9.74303E-03		9.12E-02	9.28E-01	1.46E-02	
224	2.44889E-03	1.99E+00	2.54E-02	9.28E-01	8.15E-03	0.848
896	6.08651E-04	2.00E+00	6.70E-03	9.95E-01	4.26E-03	0.936
3584	1.52175E-04	1.99E+00	1.73E-03	9.99E-01	2.17E-03	0.967
14336	3.81026E-05	1.99E+00	4.36E-04	1.00E+00	1.10E-03	0.983

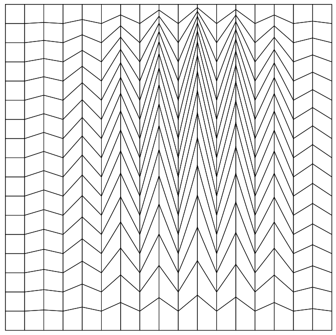
Mesh 1 - regular triangular mesh.

nunkw	erl2	umin	umax	ergrad
289	2.68581E-03	1.26800E-02	1.0020E+00	2.81E-02

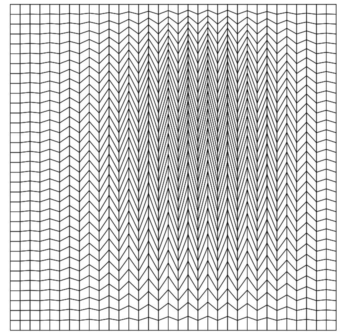
Mesh 4.1 - distorted quadrangular mesh.

nunkw	erl2	umin	umax	ergrad
1089	7.60982E-04	3.48999E-03	1.0007E+00	1.29E-02

Mesh 4.2 - distorted quadrangular mesh.



Mesh 4.1 - distorted quadrangular mesh



Mesh 4.2 - distorted quadrangular mesh

Test 1.2 The exact solution and the source term f satisfy:

$$\begin{cases} u_{\text{ana}}(x,y) = sin((1-x)(1-y)) + (1-x)^3(1-y)^2, \\ f(x,y) = -\nabla.(\Lambda\nabla u_{\text{ana}}). \end{cases}$$

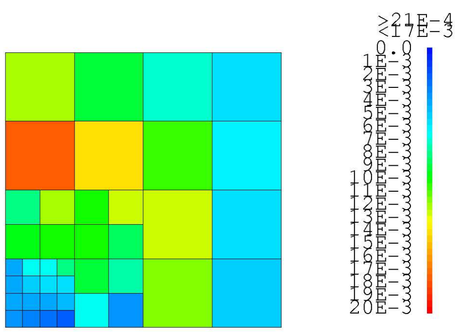
nunkw	erl2	ratiol2	umin	umax	ergrad	ratiograd
56	2.25334E-03		7.16810E-03	1.3786	1.56E-03	
224	6.03417E-04	1.90E+00	1.77495E-03	1.5973	9.52E-04	0.718
896	1.54969E-04	1.96E+00	4.42261E-04	1.7160	5.44E-04	0.808
3584	3.91813E-05	1.98E+00	1.10442E-04	1.7779	2.93E-04	0.890
14336	9.84396E-06	1.99E+00	2.75983E-05	1.8095	1.53E-04	0.938

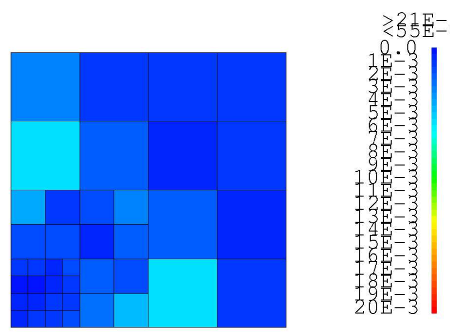
Mesh 1 - regular triangular mesh.

nunkw	erl2	ratiol2	ergrad	ratiograd
40	5.41026E-03		2.43e-02	
160	1.29132E-03	2.06E+00	1.35E-02	0.848
640	3.06998E-04	2.07E+00	7.12E-03	0.926
2560	7.43874E-05	2.04E+00	3.65E-03	0.964
10240	1.82906E-05	2.02E+00	1.84E-03	0.982

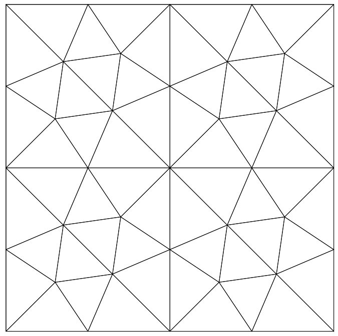
Mesh 3 - locally refined nonconforming rectangular mesh.

Error between the exact solution and the computed solution..

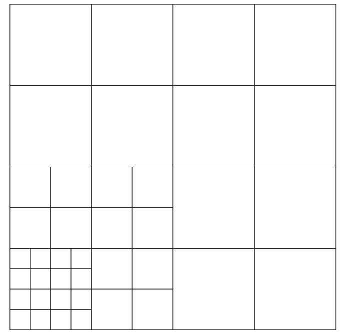




Left: Result of the MPFA scheme [1]. Right: Result of the FECC scheme.



Mesh 1 - regular triangular mesh



Mesh3-locally refined nonconforming rectangular mesh

Test 2: Heterogeneous rotating anisotropy

The tensor Λ satisfies the equation:

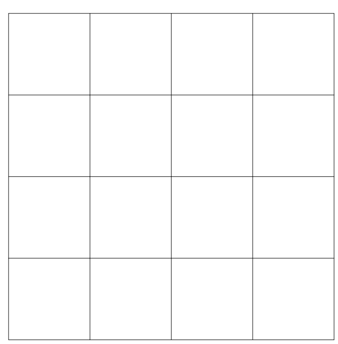
$$\Lambda = \frac{1}{(x^2+y^2)} \begin{pmatrix} 10^{-3}x^2 + y^2 & \left(10^{-3}-1\right)xy \\ \left(10^{-3}-1\right)xy & x^2 + 10^{-3}y^2 \end{pmatrix}.$$

We define the exact solution and the source term f as:

$$\begin{cases} \mathrm{u_{ana}(x,y) = \sin(\pi x)\,\sin(\pi y)} \quad \text{in} \quad (0,1)\times(0,1)\,, \\ \mathrm{u_{ana}(x,y)} = 0 \text{ on the boundary of } [0,1]\times[0,1], \\ f(x,y) = -\nabla.\left(\Lambda\nabla u_{\text{ana}}\right). \end{cases}$$

nunkw	erl2	ratiol2	umin	umax	ergrad	ratiograd
16	7.02265E-02		1.35E-01	9.34E-01	9.04E-02	
64	1.67141E-02	2.07E+00	3.68E-02	9.8E-01	5.03E-02	0.770
256	4.25124E-03	1.97E+00	9.5E-03	9.94E-01	2.80E-02	0.919
1024	1.09645E-03	1.95E+00	2.4E-03	9.98E-01	1.43E-02	0.970
4096	2.81843E-04	1.96E+00	6.02E-04	9.99E-01	7.21E-03	0.988

Mesh 2 - uniform rectangular mesh.



Mesh 2 - uniform rectangular mesh

We consider the problem:

$$\begin{cases} div\left(\Lambda\nabla u\right) = div\left(\Lambda\nabla u_{\text{ana}}\right) \quad \text{in} \quad \Omega = (0,1)\times(0,1)\,, \\ u(x,y) = u_{\text{ana}}(x,y) \qquad \text{on } \partial\Omega, \end{cases}$$

for the following tests:

Test 3: Discontinuous anisotropy (see for more detail in section 4.2.1 of [12])
The analytical solution satisfies:

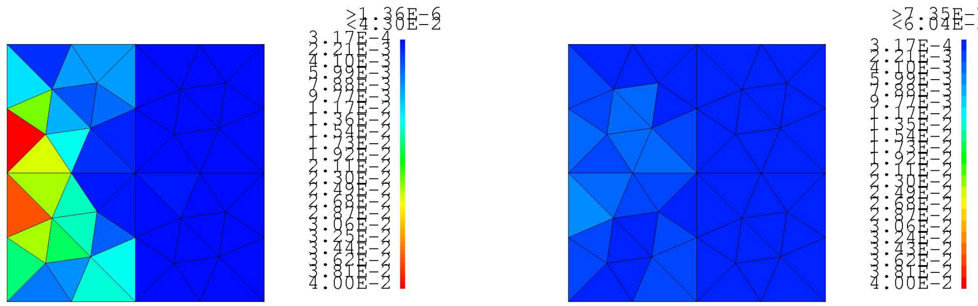
$$\begin{cases} u_{\text{ana}}(x,y) = \cos(\pi x)\sin(\pi y) \quad \text{if } x \leq 0.5, \\ u_{\text{ana}}(x,y) = 0.01\cos(\pi x)\sin(\pi y) \quad \text{if } x > 0.5, \end{cases}$$

and we consider the tensor:

$$\Lambda(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x \leq 0.5, \quad \Lambda(x,y) = \begin{pmatrix} 100 & 0 \\ 0 & 0.01 \end{pmatrix} \text{ if } x > 0.5.$$

nunkw	erl2	ratiol2	umin	umax	ergrad	ratiograd
56	5.45056E-03		-9.07E-03	9.06E-01	8.131501E-03	
224	1.37517E-03	1.98E+00	-9.76E-03	9.75E-01	6.623965E-03	0.296
896	3.44881E-04	1.99E+00	-9.93E-03	9.93E-01	3.741116E-03	0.824
3584	8.65861E-05	1.99E+00	-9.98E-03	9.98E-01	1.972594E-03	0.923
14336	2.17672E-05	1.99E+00	-9.99E-03	9.99E-01	1.011825E-03	0.963

Mesh 1 - regular triangular mesh.
Error between the exact solution and the computed solution



Left: Result of the Diamond scheme [10]. Right: Result of the FECC scheme.

Test 3.b: Discontinuous anisotropy
The analytical solution an the tensor are not changed. We use the harmonic averaging points y_σ introduced by [3] to define the dual grid:

$$y_\sigma = \frac{\lambda_L d_{K,\sigma} y_L + \lambda_K d_{L,\sigma} y_K}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} + \frac{d_{K,\sigma} d_{L,\sigma}}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} \left(\lambda_K^\sigma - \lambda_L^\sigma\right),$$

where notations are defined in Lemma 2.1 of [3], page 2.
We obtain the following numerical results with this modification:

nunkw	erl2	ratiol2	umin	umax
56	4.99875E-03		-9.07E-03	9.06E-01
224	1.28975E-03	1.95E+00	-9.76E-03	9.75E-01
896	3.32247E-04	1.95E+00	-9.93E-03	9.93E-01
3584	8.47105E-05	1.97E+00	-9.98E-03	9.98E-01
14336	2.14672E-05	1.98E+00	-9.99E-03	9.99E-01

The results are slightly more accurate than the previous results.

Test 4: Strong discontinuous anisotropy

The analytical solution is defined as follows:

$$\begin{cases} u_{\text{ana}}(x,y) = \cos(\pi x)\sin(\pi y) & \text{if } x \le 0.5, \\ u_{\text{ana}}(x,y) = 10^{-6}\cos(\pi x)\sin(\pi y) & \text{if } x > 0.5, \end{cases}$$

and we consider the tensor:

$$\Lambda(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x \le 0.5, \quad \Lambda(x,y) = \begin{pmatrix} 10^6 & 0 \\ 0 & 0.01 \end{pmatrix} \text{ if } x > 0.5.$$

nunkw	erl2	ratiol2	umin	umax	ergrad	ratiograd
56	5.45798E-03		-9.07E-07	9.06E-01	8.138566E-03	
224	1.37250E-03	1.99E+00	-9.76E-07	9.75E-01	6.624536E-03	0.296
896	3.43047E-04	2.00E+00	-9.93E-07	9.93E-01	3.741224E-03	0.824
3584	8.58622E-05	1.99E+00	-9.98E-07	9.98E-01	1.972620E-03	0.923
14336	2.14862E-05	1.99E+00	-9.99E-07	9.99E-01	1.011832E-03	0.963

Mesh 1 - regular triangular mesh.

Test 4.b: Discontinuous strong anisotropy
The analytical solution an the tensor are not changed. We also use the harmonic averaging points y_σ introduced by [3] to define the dual grid.
We obtain the following numerical results:

nunkw	erl2	ratiol2	umin	umax
56	5.03257E-03		-9.07E-07	9.06E-01
224	1.29392E-03	1.95E+00	-9.76E-07	9.75E-01
896	3.32255E-04	1.96E+00	-9.93E-07	9.93E-01
3584	8.44700E-05	1.97E+00	-9.98E-07	9.98E-01
14336	2.13100E-05	1.98E+00	-9.99E-07	9.99E-01

The results are slightly more accurate than before.

Remark 6.1: In test 3.b and test 4.b, the tensors are discontinuous on the line (d): $x = 0.5$. All the points y_σ belong to the edges σ which are common edges of the two adjacent control volumes, computed by

$$y_\sigma = \frac{\lambda_L d_{K,\sigma} y_L + \lambda_K d_{L,\sigma} y_K}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}},$$

because all the vectors λ_K^σ, λ_L^σ are equal to 0.

In test 5, we show that [3] does not provide an "acceptable" y_σ.

Test 5: Discontinuous anisotropy (case where the harmonic averaging points are not always defined)
The analytical solution satisfies:

$$u_{\text{ana}}(x,y) = \sin(\pi x)$$

and we consider the tensor:

$$\Lambda(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x \leq 0.5, \quad \Lambda(x,y) = \begin{pmatrix} 1 & 9 \\ 9 & 100 \end{pmatrix} \text{ if } x > 0.5.$$

We obtain the following numerical results with the FECC scheme:

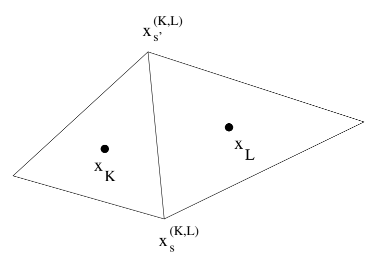
nunkw	erl2	ratiol2	umin	umax
56	2.81812E-01		1.51E-01	1.80E+00
224	2.59236E-02	3.44E+00	7.81E-02	1.10E+00
896	3.56133E-03	2.86E+00	3.92E-02	1.02E+00
3584	5.81922E-04	2.61E+00	1.96E-02	1.00E+00
14336	1.43293E-04	2.02E+00	9.81E-03	1.00E+00

Here, we can construct the FECC scheme because the dual mesh is defined. Initially, the primary mesh points were located at the barycenter of each triangle cell. We chose to slightly move them such that the hypothesis 3.1 is satisfied for any edge of the primary grid.

We give here the coordinates of a few y_σ for the coarse grid.

nunkw	K	L	$x_s^{(K,L)}$	$x_{s'}^{(K,L)}$	y_σ
56	6	18	$(0.5, 0)$	$(0.5, 0.25)$	$(0.5, -0.105128205)$
	14	24	$(0.5, 0.25)$	$(0.5, 0.5)$	$(0.5, 0.120512821)$

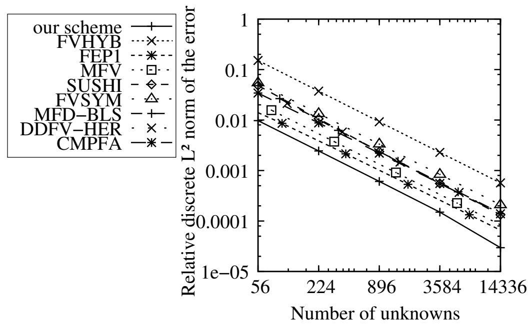
The scheme of [3] is not defined, because there are some harmonic averaging points y_σ which are outside the edges σ.



In this figure, the edge σ which is a common edge between K and L, has two vertices $x_s^{(K,L)}$ and $x_{s'}^{(K,L)}$.
We see here another difference between the FECC scheme and the scheme of [3].

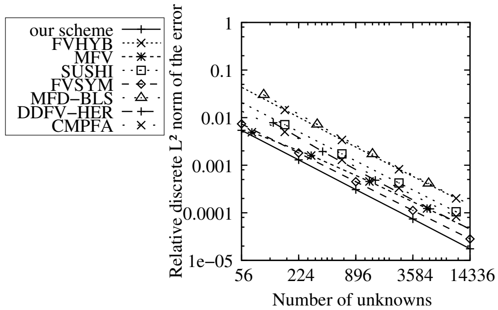
7 Comments on the results

a) In test 1.1, for triangular cells, where the mesh and the solution are regular, we obtain a second order convergence in the L^2 norm and an order close to 1 for the gradient.

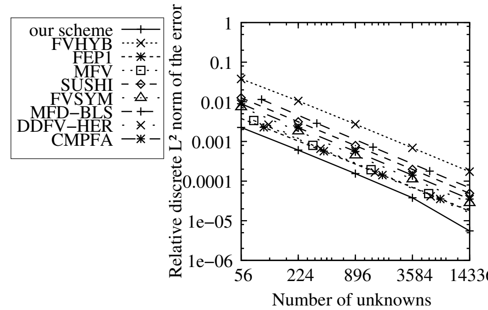


Mesh 1 – regular triangular mesh

b) In test 1.2, we obtain an order of convergence in the L^2 norm close to 2 for regular triangular meshes and locally refined nonconforming rectangular meshes. The order of convergence of the gradient tends toward 1 for regular triangular meshes and locally refined nonconforming rectangular meshes.

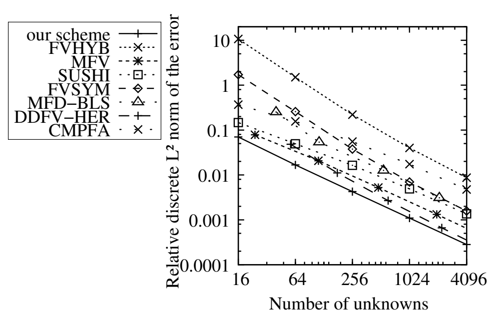


Mesh 3 – Locally refined nonconforming rectangular mesh



Mesh 1 – regular triangular mesh

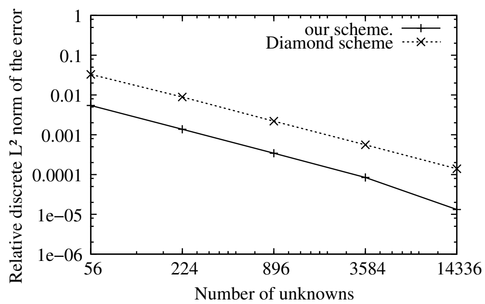
c) In test 2, for uniform rectangular meshes where Λ is an heterogeneous tensor, we obtain an order of convergence near to 2 in the L^2 norm and the order of convergence of the gradient tends toward 1.



Mesh 2 – uniform rectangular mesh

From the graphs which describe the number of unknowns and relative discrete L^2 norm of the error, we see that the errors of the FECC scheme are less important than the errors of [4], [6], [16], [23], [24], [22], [20], [15] and the orders for the gradient of the scheme are close to those of the Galerkin finite element method (see [6]).

d) In test 3, for regular triangular meshes where Λ is discontinuous, we obtain an order of convergence near to 2 in the L^2 norm and the order of convergence for the gradient tends toward 1. With the same number of unknowns, the errors of the scheme in the L^2 norm are less than the errors of [10].



Mesh 1 – regular triangular mesh

e) In test 4, for regular triangular meshes where Λ is an heterogeneous tensor with a strong anisotropy, we obtain an order of convergence in the L^2 norm near to 2 and the order of convergence for the gradient also tends toward 1.

We compare the FECC scheme with the following methods:
Cell centered schemes:

- CMPFA: *Compact-stencil MPFA method for heterogeneous highly anisotropic second-order elliptic problems*, [23].
- FVHYB: *A symmetric finite volume scheme for anisotropic heterogeneous second-order elliptic problems*, [4].
- FVSYM: *Numerical results with two cell-centered finite volume schemes for heterogeneous anisotropic diffusion operators*, [24, 25].
- SUSHI: *A scheme using stabilization and Hybrid Interfaces for anisotropic heterogeneous diffusion problems*, [16].

Discrete duality finite volume schemes:

- DDFV-HER: *Numerical experiments with the DDFV method*, [22].

Finite elements schemes:

- FEP1: *A Galerkin finite element solution*, [6].

Mixed or hybrid methods:

- MFD-BLS: *Mimetic finite difference method*, [20].
- MFV: *Use of mixed finite volume method*, [15].

8 Conclusion

We constructed a cell-centered scheme for diffusion problem on general meshes. With light assumptions, the matrix which is associated to the scheme is symmetric positive definite. This allows us to use efficient methods to solve the system of linear equations. Moreover, the scheme is locally conservative. The comparison with standard schemes shows that our scheme is very accurate in L^2 norm. Moreover, we proved that the FECC scheme is convergent in the anisotropic discontinuous cases.

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